Rabbits and Sequences Masterclass: Session Script

The Royal Institution
Science Lives Here
Masterclass network

This icon means there's a slide, or slides in the presentation to accompany this line of the script.

This icon indicates the students will have an activity to do, or something to write.

Introduction (10 minutes)

Welcome to today's Masterclass. Today we will be looking at sequences, including sequences of rabbits.
[If you would like them to start working on the activities, replace slide 1 with slide 8 – see the script notes for that slide]

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[This next section can be skipped if this masterclass is not the first one in the series.]

These masterclasses are organised by the Royal Institution. Has anyone heard of the Royal Institution before? It was founded in 1799, and has always been about letting everyone have access to science - organising masterclasses and lectures, including the Christmas lectures.

2 – 7

Many famous discoveries have been made in the Faraday building where they are based, including 10 chemical elements. Michael Faraday, who the building is named after, did work there on electricity and optics, which we all use every day.

The Ri are perhaps most famous for Christmas Lectures for young people, which have taken place since 1825. They have been televised for several decades, and many past series are available on the Ri website. The Masterclass programme was born out of the Christmas lectures delivered by Christopher Zeeman in 1978, which were the very first ones on mathematics!

There are many opportunities for you to visit the Ri building in London, and see the historic rooms for yourself. There is a small museum too. There are lots of talks and holiday events for young people: all the details are on the Ri's website.

Sequences and rules (50 minutes)

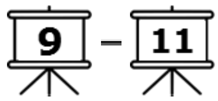
In this Masterclass session, we'll be thinking about sequences of numbers. To start with, there's an activity for you to work on - can you complete these sequences by working out what comes next, and explain why you think your answer is correct?

[Worksheet 01 – "Find the next number in the sequence"]

For those who finish quickly: **Now come up with your own examples of sequences.**

You've been completing the sequences on the sheet – let's go through and see some possible answers. If you have a different answer to what is on the board, it might still be right – you just have to have a good reason which works for that sequence.

- **Put your hands up if you have an answer to each question. Can you explain why these answers are correct – what is your reason?**



[Go through the answers each sequence at a time; discuss any different answers and the reasons, and if the reasons work, those answers are fine. If the sequences are familiar, e.g. triangle and square numbers, as the students about this – did they work out reasons, or just use the next numbers they remembered?]

Each of the sequences has a rule which tells us how to find the next number in the sequence. The numbers we already have follow this rule, and so will any future numbers.

- **For those of you who have made up your own sequences – what are they? Make sure you explain why what you've written is a sequence - can you come up with a rule that explains how to find the next number?**

[Write the sequences and rules on the board]

Many of the rules we have come up with have told us how to get from one number in the sequence to the next. This is called the **term-to-term rule**.

- **What are the next two numbers in this example?**
- **What is the rule?** [Click to show the answers]

We usually want to try to write the rule for each sequence mathematically, and we use some mathematical notation to do this. This is a way of writing things that other mathematicians will understand. 13
Let's start with looking at what position each number in the sequence has. We can match each number in the sequence with its position and we can label the position numbers with "n". n can change according to what position we're looking for – when n=1, that's the first position, when n=2, it's the second position, and so on.

Each number in a sequence is called a term. If we label the sequence "a", we can label each term in the sequence too. The first term will be "a_1", a with a tiny 1 next to it; the second term will be "a_2", a with a tiny 2 next to it, and so on. The tiny numbers at the bottom of the letter are called "subscripts". They are a good way of labelling the terms, because it tells us which term we are looking for and we know it is a label, not a number, if we're doing sums involving the terms in the sequence. 14

If we don't know, or don't mind, which particular term we are looking for we can label it "a_n" – like before, the n can be any number we want it to, so this label can point to any of the terms in the sequence. We call this the **nth term**. 15

So, when n=1, $a_n = a_1 = 5$; n=2, $a_n = a_2 = 10$; and n=3, $a_n = a_3 = 15$ 16
[Check the students understand the labelling - get them to give the answers to the numbers in the sequence]

Because we can point to the nth term, we can also point to the term before and the term after. The term before can be labelled with "a_{n-1}" and the term after can be labelled with "a_{n+1}" 17

So, as an example, if n=2, we know that a_n is a_2 which is 10. 18
n-1 = 2-1 = 1, so $a_{n-1} = a_{(2-1)} = a_1$ and $a_1 = 5$.
n+1 = 2+1 = 3, so $a_{n+1} = a_{(2+1)} = a_3$ and $a_3 = 15$.
Remember that the subscripts are a label, not really numbers – the calculations we are doing are just to help us point to the correct term of the sequence.

We can use this notation to find a mathematical way to write the rules for the sequences on the sheet. We will need to know how they start, so we will need a value for a_1, and we will need to know how to get to the next term from the term before – our rule will look like "a_n = …" something including "a_{n-1}". This means we will know what a_n will be if we know what a_{n-1} is. 19

Go through each of the sequences on your sheet and see if you write a mathematical rule like this for each one. Can you do this for the sequences you made up too? [Students might need a lot of support – see session leader & helper notes.]

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- **What is the first term in the sequence, a_1? What is the mathematical rule?**
- **Why is it important to know the first term in the sequence?**
- *Optional:* **Which ones did you find difficult and why?**

[You might need to discuss that it's OK to have different rules for odd and even terms; and that it's easier to find a different kind of rule for the last sequence, the square numbers. If they have any of their own sequences, go through these on the board – you may need to allocate more time to this.]

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The rules we've written so far are ways to write the next number, if you know what the one before was. But for some of these sequences, we can write a rule which tells us how to find any number in the sequence, even if we don't know the ones before it. Instead of including "a_{n-1}" in our rule, we would have an equation involving "n" – the position number. We call these rules **position to term rules**.

For example, we saw this sequence before:

- **How do we get to a_1 from the position number, 1?**
- **How do we get to a_2 from the position number 2?**
- **How do we get to a_3 from the position number 3?**
- **What is the rule in terms of "n"?**

[Ask the questions before clicking to show the equations. show them that 5xn can be written as 5n.]

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- **So, our rule is $a_n = 5n$ – what is the 100th term?**

25

We have seen this sequence before and came up with a term-to-term rule.

- **What is the position to term rule for this sequence?**
- **Why is this the rule?** [This is the square numbers so the n^{th} number in this sequence is $n \times n$ or n^2].

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- **What is the 100th term?**

If time: Have a go at writing the position to term rules for the sequences on your sheet. For some sequences, both kinds of rule are easy to write, but for others it gets a bit more complicated!
[Answers are on the background information sheet.]

Leonardo’s Rabbits (25 minutes)

Next we’re going to look at a way to use a sequence of numbers to describe something that’s happening in the real world. A mathematician called **Leonardo of Pisa** worked out a way to study populations of rabbits, using a mathematical sequence. Here are his rules for how rabbit populations work - you can see they’re a simplification of how this works in the real world, but sometimes in maths it’s useful to make a problem simpler first so you can understand it.

- **Which of these rules are realistic, and which are simplifications?**

After each one, we can discuss whether this is what really happens (from our knowledge of real animals).

- Rabbits always live in pairs
- Once they’re born, rabbits carry on living forever
- Each pair of baby rabbits takes **one month** to become adults
- Rabbits can only have babies once they have become adults
- Each month, every pair of adult rabbits has **exactly one pair of baby rabbits**

[Hand out Worksheet 02 – “Rabbit rules 1”]

Now we can use these rules to work out the sequence we get for how many pairs of rabbits there are in total, each month for a year. Remember that a pair of rabbits takes a month to become adult rabbits, and they won’t have any baby rabbits during the month they’re growing into adults - only once they are adults already. Let’s work out the first few rows together.

Use the worksheet to work out the number of rabbits. We’ve started with one pair of baby rabbits, and see what happens over the rest of the year.

Once you think you’ve filled the sheet, compare with your neighbour. If you didn’t get the same answers, can you see why? Make sure you’ve followed the rules correctly.

- **Can you find a rule for how the total number of rabbit pairs changes each month?**

[Worksheet 03 – “Rabbit rules 2” – if time]
Try starting with different numbers of rabbits. We have extra copies of this grid so you can see what happens for different starting numbers - try

using one pair of adults and two pairs of babies, or come up with your own combinations.

Let’s discuss our results together. 33

- **Does the number of rabbits always keep getting bigger?**
- **Does the rule for getting the next number of rabbits change?**

[Go through the answers for starting with one pair of baby rabbits on the board]

- **Why do these rules for rabbits lead to sequences like this?** 34

Each generation, new rabbits are produced - and this number is added to the number of rabbits in the previous generation. The number of new pairs will be the same as the number of adult pairs in the previous generation, as each adult pair produces one pair of babies. 35

This is added to the number of rabbits that were adults already, and all the baby rabbits that have just grown up. 36

But every rabbit that’s old enough to have babies will have come from either a rabbit pair that was already there, or one that’s just matured, so the number of new babies will be the same as the number of rabbits in the generation before last. 37

The sequence of numbers shown on the slide is what you should get starting with one pair of baby rabbits, and the numbers for each type of rabbits are shown. This is called the **Fibonacci Sequence**, and there’s a rule for how to find each number - this time instead of using the number before, like the sequences we’ve already seen, it uses the two numbers before, and you add them together. 38

You may have heard of this sequence before. It’s named after **Leonardo de Fibonacci**, who was also called Leonardo of Pisa. You might have found that if you start with different pairs of numbers, sometimes you get to the same sequence, and sometimes you get a different one - but the rule for finding the total number of rabbits in each generation is always the same. 39

If you start from 2 baby pairs and 1 adult pair, you’ll get this sequence - these are called **Lucas Numbers**, and they’re named after another mathematician called Edouard Lucas. They are a different set of numbers, but each one is still the sum of the two numbers before. 40

Fibonacci was the first person to use this to describe the way populations of rabbits might grow. Of course, like any mathematical model, this isn’t strictly the way things work in the real world! The number of rabbits in our sequence will always carry on increasing, which it doesn't in the real world - or else we'd be overrun with millions of rabbits! So to get a more accurate model, you might need to find out how long a rabbit usually lives (it's not forever!), and build that in to your model. But this model does produce the Fibonacci sequence, which is why it's interesting.

Coins and Combinations (25 minutes)

Now we're going to try something different. You each have a collection of coins, and on the screen you can see a challenge - how many ways can you make a line of coins that adds up to the same total? Let's do the first few together.

- If we want a total of 0p, there's only one way to make a line of coins - by not having any coins!

- For a total of 1p, there's also only one way to make a line of coins - just place 1p on its own.

- If I wanted to make a line of coins that total 2p, there's two ways to do this - can anyone suggest what they might be? 1 2p coin, or 2 1p coins.

- And if I wanted to make 3p, what would my options be?
  I could use 3 1p coins in a line, or I could place 1 1p and 1 2p. But I have a third option - because we're going to count it as a different sequence if I have 1p 2p or 2p 1p. So there are three possibilities here.

Now you can carry on and find the ways to make a line that adds up to 4p, 5p and so on. Use the squared paper, and find a way to record your results systematically - so it's clear what you've found.

When you've got some answers, check with your neighbour to see if you have the same, and if not, see if you have missed any.

- **Can you see a pattern in the number of ways to make a line?**

[Go through answers on the board]

You might have noticed that this pattern is familiar - it's the Fibonacci sequence again! If you spotted this, I hope you carried on working out all the different arrangements, and didn't just fill in the numbers to carry on the sequence!

When you're working with sequences of numbers like this, sometimes you'll find that the same numbers crop up in different places. But if you just have a sequence of numbers without a rule, there's no way to know if they will carry on being the same, or if something will change and later on the sequence will be different. You need to be careful! In mathematics, seeing patterns is a really useful way to start, but it's great to be able to prove something as well.

For the Fibonacci sequence, we have a rule - each number is the sum of the two previous numbers. But we don't know that the lines of coins we're making will follow this, unless we can see a reason why that gives the same rule.

- **Look at your sequences, and discuss why the number of possible lines might follow this same rule. Do any of the lines of coins you've made share anything in common with other lines of coins? Compare the lines that add up to 5p with the lines that add up to 4p and 3p.**

Can anyone explain why this pattern occurs? Think about the ways to make 5p you have in your list, and compare them to the ways to make 4p and the ways to make 3p. Each of the 5p lines of coins will be the same as one of the 4p or 3p lines of coins, with an extra 1p or 2p added at the end. Each time we add to a sequence, we can either add 1p or 2p (as those are the only coins we have).

Each time we go to the next number for the total, we are increasing the number of combinations in eactly the same way as we increased the number of rabbits. So these two sequences will always be the same, as they follow the same rule!

End of session – recap

Remove if you are including the section on prime numbers – instead use the recap at the end

In this session we've looked at sequences of numbers; we have talked about how important the starting numbers of a sequence are; we studied how the number of rabbits changes and how you can use a mathematical sequence to model it; and we made lines of coins to find the same sequence again!

When you see a sequence of numbers in the future, look for patterns and see if you can find a rule for the next number - but more importantly, try to understand why that rule will always work! See if you can find the Fibonacci or Lucas numbers in the world around you – look at spirals on the numbers of petals in flowers, and see if the world is a lot more mathematical than you first thought.

Prime Numbers (15-20 minutes)

Another sequence of numbers you might be familiar with is the **prime numbers**.

- **Can anyone tell me the definition of a prime number?**

A prime number is one which can only be divided by 1 or itself, and we don't include 1 as a prime number. Suppose we want to make a list of all the prime numbers.

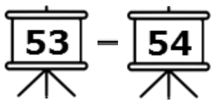
- **How could we make a list of all the primes?**

To decide whether a number is prime you need to check whether it divides evenly - and for small numbers this is easy, but as the numbers get bigger there's a lot of things to check. One thing that's slightly easier is to make a list of all the numbers which aren't prime! If a number is not prime, it might be divisible by 2, or divisible by 3, or 5, or something else. So if we take a list of all the numbers and cross out everything that divides by 2, then everything that divides by 3, and so on, we'll be left with the numbers that are prime.

Using this grid, we're going to make a kind of 'number sieve' - this is named after the mathematician Eratosthenes, and it's literally called the **Sieve of Eratosthenes**.

Use this 10 by 10 grid to work out which numbers are prime - start with 2, and don't cross out 2, but cross out everything that's a multiple of 2, by counting on. From 2, skip the number 3 then cross out the number 4, skip the 5 and cross out the 6 and so on. Then go back and do the same for 3 - don't cross out 3, but cross out 6 (although it will already be crossed out!), 9 etc

Once you've gone beyond 7, you'll find you have found all the primes less than 100. Now, use these other grids to look for patterns - circle the prime numbers on these other grids and see if you can find any patterns in them.



Here's what the primes look like in those two grids. This one on the right is really interesting - you should find that in the grid that's 6 squares wide, all the prime numbers fall into these two columns.

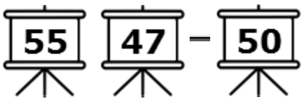
- **Can you see why this might happen? What's true about the numbers in columns 2, 3, 4 and 6?**

All the numbers in column 2, 4 and 6 are divisible by 2, and those in columns 3 or 6 are divisible by 3, so they can't be prime. This means all the prime numbers will land in columns 1 or 5.

So we can see some patterns in the prime numbers - but in general there's no rule that tells us where the next prime number will be! It's one of the biggest unsolved mysteries in mathematics, as prime numbers are very important, and it would be very useful if we could find a rule for them.

Prime numbers are the building blocks that all the other numbers are made from, and they carry on forever. They are very important in lots of different areas of maths, and they're even used to make sure data you send and receive on the internet is secure, so people consider working out a rule for them very important. In fact, if you're able to work out the pattern in the prime numbers, there's a prize of $1 million available for the first person to do it, as it's connected to one of the Millennium Prize problems.

End of session - recap



In this session we've looked at sequences of numbers; we studied how the number of rabbits changes and how you can use a mathematical sequence to model it; we made lines of coins and found the same sequence again; and we looked for patterns in the prime numbers - but it might take a bit more work before we find the rule!

When you see a sequence of numbers in the future, look for patterns and see if you can find a rule for the next number - but more importantly, try to understand why that rule will always work! See if you can find the Fibonacci or Lucas numbers in the world around you – look at spirals on the numbers of petals in flowers, and see if the world is a lot more mathematical than you first thought.