Rabbits and Sequences Masterclass:
Extra Background Information

The Royal Institution
Science Lives Here
Masterclass network

A sequence, in mathematics, is a string of objects, like numbers, that follow a particular pattern. The individual elements in a sequence are called **terms**. A sequence may contain the same number twice, and it doesn't always have to carry on getting bigger or smaller. The order of the terms in the sequence is important, and the sequence can be finite (finish after a certain number of terms) or go on forever. Sequences are useful in a number of mathematical disciplines for studying patterns, shapes, and other mathematical structures.

In this session, we discuss sequences of numbers, and how we can give rules for the terms of the sequence. Two possible ways to define a sequence are:

- to give a way to calculate each term in the sequence using the previous term (using the notation a_n, to mean the n^{th} term of the sequence - we can define a_n in terms of a_{n-1}), sometimes called a **recurrence relation** or a term-to-term rule,
- or to define the n^{th} term in the sequence, usually in terms of n, sometimes called a position-to-term rule.

In the case of a recurrence relation, it's also important to give the first term of the sequence, so that each subsequent term can be calculated from it. For an n^{th} term formula, the first term is the one obtained by substituting n=1 into the formula.

In the session, we give a list of examples of sequences, and give both types of formulae for some - but with others, the formulae are more complex (but still possible to find!) You'll notice some of the examples below are split into two cases, for odd and even numbers. Students in this workshop may come up with their own sequences, and you may not be able to find an obvious formula for them on the spot - but as long as they can describe their rule in words, and it will always give a valid answer, that's fine.

Sequence	Recurrence relation	n^{th} term formula
5, 10, 15, 20, 25, 30 Multiples of 5	$a_1 = 5$, $a_n = a_{n-1} + 5$	$a_n = 5n$
1, 3, 5, 7, 9, 11, 13 Odd numbers	$a_1 = 1$, $a_n = a_{n-1} + 2$	$a_n = 1 + 2(n-1)$
2, 4, 8, 16, 32, 64 Powers of two	$a_1 = 2$, $a_n = 2 \times a_{n-1}$	$a_n = 2^n$
1, 4, 8, 11, 15, 18, 22, 25 'Add three then add four'	$a_1 = 2$, $a_n = a_{n-1}+3$ (n even), $a_n = a_{n-1}+4$ (n odd)	$a_n = 1+7((n-1)/2)$ (n even), $a_n = 4 + 7((n/2)-1)$ (n odd)
1, 3, 6, 10, 15, 21, 28 Triangular Numbers	$a_1 = 1$, $a_n = a_{n-1} + n$	$a_n = (n \times (n-1)) / 2$
4, 5, 3, 6, 2, 7, 1, 8 Add then subtract successive numbers	$a_1 = 2$, $a_n = a_{n-1} + (n-1)$ (n even), $a_n = a_{n-1} - (n-1)$ (n odd)	$a_n = 4 + \sum_{i=0}^{n-1}(-1)^i(-i)$ $= 4 + (-1)^1(-1) + (-1)^2(-2) + \ldots + (-1)^{(n-1)}(-(n-1))$
1, 4, 9, 16, 25, 36 Square numbers	$a_1 = 1$, $a_n = a_{n-1} + (2n+1)$	$a_n = n^2$

Fibonacci Numbers

One sequence that's studied in this workshop is the Fibonacci sequence, defined as:

$$a_1 = 1,\ a_2 = 1,\ a_n = a_{n-1} + a_{n-2}$$

Each entry in the sequence is the sum of the two previous entries. This means to define the sequence completely, the first two terms must be given. Different pairs of starting terms can result in different sequences - for example, starting with $a_1 = 2$, $a_2 =1$ gives the Lucas numbers. While the resulting sequences are different for different starting conditions, the relation between subsequent terms is always the same.

Fibonacci numbers occur in many places - as can be seen in this workshop, calculating the number of ways to arrange 1p and 2p coins produces Fibonacci numbers, as does a simple model for rabbit populations. They're also connected to Pascal's triangle, and to the Golden Ratio. They're so common in mathematics, a whole journal called Fibonacci Quarterly is dedicated to new discoveries connected to Fibonacci numbers. They also crop up in divisibility algorithms, project planning methods, search algorithms, and generating random numbers.

The Fibonacci sequence is named after Leonardo de Fibonacci, who lived approximately 1175 – 1250, and was sometimes called Leonardo of Pisa. While Fibonacci wasn't the first person to discover this sequence, he was the first to introduce the idea to Western mathematics, in his book in 1202, which was called Liber Abaci ('Book of Calculation'). In this book he also popularised the system of Hindu-Arabic numerals we use today, including the use of the digit 0 and place value in decimals.

Fibonacci used the sequence as an example in his manuscript - he suggested a model for populations of rabbits, in which each pair of rabbits produces one pair of baby rabbits in a given time period, and the baby rabbits take the same time to mature to be able to produce their own offspring. The model isn't realistic, as it assumes rabbits live forever, and only produce one pair of babies - it also doesn't account for rabbits changing partners or populations migrating, so it's a very simplified model, but it results in the Fibonacci sequence.

Successive pairs of terms in the Fibonacci sequence have ratios which converge to the Golden Ratio, Φ (phi) which equals $(\sqrt{5}+1)/2$. That is, if you take a pair of terms that follow each other and divide the smaller by the larger, the further you go down the sequence the closer your answer will be to Φ.

Since the ratio between one mile and one kilometre is close to the Golden ratio, this means that if you know a distance in miles and want to find it in kilometres, and your number of miles is in the Fibonacci sequence, the number of kilometres will be roughly the next Fibonacci number - for example, 3 miles is around 5km, and 8 miles is around 13km.

Adding together the 'shallow diagonals' in Pascal's triangle (lines which run at a shallow slope across the triangle, moving across one and up one to pass through numbers) will result in the Fibonacci sequence.

Fibonacci numbers also occur in nature - often, counting the number of spirals on a pine cone or sunflower seed head will result in a Fibonacci number. This is because the spacing of the seeds is optimal when it's least likely to overlap with itself, so the Golden ratio is useful in making the gap between seeds least likely to be a fraction of a whole turn - if the seeds were placed ¼ turn apart, every fourth seed would overlap, but with seeds $1/\Phi$ of a turn apart, it will be a long time before a seed lands in the same place, and this means the number of turns will be a Fibonacci number. Of course, every plant is different, and natural variation means this sometimes doesn't work.

Prime numbers

A prime number is a number which can only be divided evenly by one and itself. We exclude the number 1 from the set of prime numbers, and a whole number greater than 1 which is not prime is called a composite number. To check whether a number is prime, the simplest way is to test by dividing by all the prime numbers smaller than the square root of the number (for example, to check 101 is prime you'd need to check all the primes smaller than $\sqrt{101} \sim 10$). This process can be long and laborious for large prime candidates, and other methods have been devised for checking primality which take less calculation time for large numbers - but it's still not a simple check.

The method used in this workshop to find the prime numbers is called the Sieve of Eratosthenes - it's attributed to Eratosthenes of Cyrene, a Greek scholar from the third century BC, who is best known for being the first person to calculate the circumference of the Earth, and the tilt of its axis, and for inventing the idea of a Leap day. The "sieve" involves writing all the whole numbers then going through and crossing out all the multiples of 2 except for 2, then all the multiples of 3 except for 3, and so on. The number of numbers crossed out on each successive sweep generally gets smaller - for example, when you come to cross out multiples of any even number beyond 2, you'll find they're already crossed out, and in fact only multiples of larger prime numbers will be left in. To cross out all the non-prime numbers below 100, you only need to cross out multiples up to multiples of 7, as there are no prime numbers bigger than 7 and smaller than $\sqrt{100} = 10$.

There are infinitely many prime numbers (demonstrated by Euclid in around 300BC), but while many facts about prime numbers are known, they're one of the biggest sources of unanswered questions in mathematics. Conjectures such as the Twin Prime Conjecture and Goldbach's Conjecture are related to prime numbers, and remain unsolved. There's no known pattern in the prime numbers, or a simple way to find the next prime number, and the Riemann Hypothesis is another related mathematical idea which would help us to understand the pattern a little better - if only we could prove it is correct!

The Riemann Hypothesis is one of the seven Millennium Prize Problems, given by the Clay Mathematics Institute in the year 2000. The list of problems includes questions from all different areas of maths, including fluid dynamics, computability, topology and number theory. Each has a prize of $1m available to whoever solves it first - and while one of the seven problems, the Poincaré Conjecture - has been solved, the Riemann Hypothesis remains open.

Prime numbers are hugely important in maths - they're the building blocks of all the other numbers. Every number can be uniquely written as a product of the prime numbers that make it up, and this property is known as the Fundamental Theorem of Arithmetic. Prime numbers are used in cryptography and data security, and to make sure information is transmitted correctly. Cicadas, which are insects a little like crickets, have prime-numbered years in their life cycle, which means they're less likely to hatch at the same time as their predators.