# **Sierpiński’s Triangle**

In this series of activities, pupils have an opportunity to explore patterns in maths, starting by considering observed patterns in number sequences.

Through playing ‘the chaos game’, pupils will be introduced to the concept of fractals, with particular focus on the Sierpiński (Pronounce see-er-pin-ski) triangle. This structure is investigated and later revisited through studying the pattern of odd and even numbers in Pascal’s triangle.

For students who have already encountered Pascal’s triangle in another Masterclass, or for in cases where time is limited, a shorter version of the odd/even activity is included. There are also extension activities discussing the properties of fractals.

**Overview of activities:**

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| **Slides & Time** | **Overview** |
| Slides 2-7  (5 minutes)  (Total: 5 mins) | Intro to Ri (If not done at another Masterclass) |
| Slides 8-10  (10 minutes)  (Total: 15 mins) | Introductory activity - ‘what comes next?’ followed by discussion |
| Slides 11-33  (25 minutes)  (Total: 40 mins) | Chaos game activity and follow-up discussion |
| Slides 34-43  (25 minutes)  (Total: 65 mins) | Sierpiński triangle construction |
| BREAK  (10 minutes)  (Total: 75 mins) | Drinks and biscuits and comfort break |
| Slides 44-45  (15 minutes)  (Total: 90 mins) | Fractal properties discussion |
| Slides 46-65  (10 minutes)  (Total: 100 mins) | Construct Pascal’s triangle activity (can be omitted) |
| Slides 66-72  (10 minutes)  (Total: 110 mins) | Investigate fractal patterns in Pascal’s triangle and why this arises |
| Slides 73-79  (10 minutes)  (Total: 120 mins) | Recap and end of session feedback/takeaway |

**General Masterclass resources needed:**

* Register of children
* Consent forms and emergency information to hand
* Stickers and markers for name badges
* Adult register
* Ri child protection policy
* Paper and pencils/whiteboards for workings
* 2 different coloured post it note pads
* Drinks and biscuits

**Specific resources needed:**

Data projector/computer to display slides and internet access to display online applets

* Chaos game applet: [bit.ly/sierpinski-chaos](http://bit.ly/sierpinski-chaos)
* Sierpiński midpoint app: [bit.ly/sierpinski-midpoint](http://bit.ly/sierpinski-midpoint)

**Chaos Game activity**:

* **Worksheet 01:** OHP transparencies printed/traced with triangle dots (1 per pair of students)
* 1 six-sided die per pair of students
* Pens to write on the transparencies (dry wipe or OHP pens)
* 30cm ruler per pair of students
* Overhead projector or backlit visualiser to display the finished transparencies in a stack (looking through them all)

**Sierpiński triangle construction**:

* Isometric paper (grid of lines, not dots; printable version included in **Worksheet 07**) - 1 sheet per student
* Coloured pencils or pens
* **Worksheet 02**: Table completion worksheet for Fractal properties discussion (1 per student)
* For long version of Pascal’s triangle activity:
  + **Worksheet 03**: Small blank Pascal’s triangle worksheet - 9 rows (1 per student)
  + **Worksheet 04**: Addition rules worksheet for Pascal’s triangle activity (1 per student)
  + **Worksheet 05**: Large blank Pascal’s triangle worksheet - 32 rows (1 per student)
* For short version of Pascal’s triangle activity:
* **Worksheet 06:** Filled triangle worksheet (1 per student)

### **Things to prepare in advance:**

* Print worksheets, isometric graph paper and gather equipment listed above
* Print triangle dots onto OHP transparencies, or trace them from a printed version using permanent marker
* Load or download online applets and check they are working

**Support resources:**

* Helper notes: An overview of the Masterclass content and activities
* Supporting notes: Extra information and background on Sierpinski and Pascal and the specific examples used in this Masterclass
* Session script: Suggested wording for each section of the session
* Solutions to worksheets 2, 4 and 6

**Ask the Ri:**

Don’t forget to collect any questions which arise, and email them to the Masterclass team at the Royal Institution: [masterclasses@ri.ac.uk](mailto:masterclasses@ri.ac.uk)

**Feedback:**

We would very much welcome your feedback on this session. If you have time, please collect feedback from the students at the end of the Masterclass and send it through to us. We would also appreciate feedback on how you have used the session, what you think worked well and what improvements would be useful.

# **Session Guide**

In the following, some questions you can address to the students are indicated with a bullet point, and where specific slides in the presentation accompany a particular section, we’ve included which slide numbers.

We’ve included slides 2-7 at the start of the presentation so you can explain that this Masterclass session is part of a network of series all over the UK, coordinated by the Royal Institution in London (home of the Christmas Lectures). If you have done this before in your series, start at slide 8.

## **Introduction (10 minutes)**

\*ENSURE STUDENTS HAVE ROUGH PAPER/PENCIL OR MINI WHITEBOARDS/PENS\*

Introduce the session by explaining that in this session we’ll be thinking about patterns (show slide 9 with the word ‘patterns’).

* Ask the students to name an example of a pattern they’ve seen.

Acknowledge there are many different kinds of patterns - on fabric, wrapping paper or wallpaper, and in numbers, words and shapes. We’ll start this session with a pattern in numbers.

* Show slide 10 with the numbers 1, 2, 4… and ask the students to write these numbers down, then write what they think comes next.

Some students will write 7, and some will write 8 (and others may have other answers).

Acknowledge that students who wrote 7 were thinking about the gaps between the numbers, which go 1, 2, 3… and those who wrote 8 were thinking about the fact that the numbers double in size each time, so 8 would be next.

Explain that even though both of these are good answers (and any other answers which have a pattern they’re following are good too) but we can’t say for sure if any of them are ‘right’. Mathematicians spend a lot of time looking for patterns, and if we see a pattern in an unexpected place, it can be very exciting. If you try to understand why the pattern is there, that can be very illuminating and lead you to more interesting maths.

Explain that next we’re going to look at a visual pattern, made using a simple set of instructions. This is called the Chaos Game (the name is on slide 11) - this isn’t the sort of game where you win or lose, but you create something interesting.

## **Chaos Game Activity (25 minutes)**

Explain to the students this pattern will be made using “iteration” (which means repeating the same instruction over and over). Show slide12 with the word ‘iteration’

Explain the rules of the game - they’ll be working in pairs:

* Starting with a triangle (corners labelled as shown), roll the die to choose a corner to start from, as shown on slides 13-15
* Roll again to pick a second corner, and draw a line between them (slides 16-17)
* Measure this line and mark a point halfway along it (slides 18-20 have the example of rolling a 4, then a 2 and moving half way along the line which is 20cm long). Explain you don’t need to be completely exact with your measurements, as long as your point is in the middle of the line.
* Now roll again and move from the point you’ve just drawn halfway to the corner point you rolled (slides 21-24 have the example of now rolling a 6, then a 1).

Before you ask the students to start the activity, ask them to imagine what the pattern will look like - will the dots be spread evenly across the triangle? Will any dots fall outside the triangle? Will there be any places inside the triangle you won’t draw any dots?

\*GIVE OUT (PER PAIR) WORKSHEET 01 ON TRANSPARENCY, DICE, PEN & RULER\*

Give them five minutes to draw dots. Warn them to be careful with the marker pens, and not to accidentally rub off their dots if they’re using dry erase pens. The instructions are on the next slide (slide 26).

Collect in everyone’s sheets and stack them on the projector, so you can see all the dots together. Look for regions with no dots, and mark them on the top sheet - there should be a large region in the middle, and others in the corner sections.

* Ask the students to estimate how many dots they drew on each sheet and how many dots they drew altogether.

Suggest maybe this isn’t enough dots to see the pattern in detail. Load the online **Chaos Game Geogebra applet** at [bit.ly/sierpinski-chaos](http://bit.ly/sierpinski-chaos) and run it to show what happens as the number of dots increases. Then show slide 27 which has a video animating this process and adding 100 dots per second, which can be used instead of or as well as the applet.

Explain the resulting shape is called a **Sierpiński triangle** (shown and labelled on slide 28), and ask them to describe the shape. There’s a hole in the middle where no dots land, and in order to explain why, use the **Midpoint Geogebra applet** at [bit.ly/sierpinski-midpoint](http://bit.ly/sierpinski-midpoint) (slide 29)to show that if the end of your line is inside the triangle, the midpoint can only land in the smaller triangle nearest the corner you’re moving towards - as this smaller triangle is half as tall, and half as wide.

Discuss what it means that your starting point will never be inside the middle triangle - it means you’ll never have the midpoint land inside the smaller triangle either. Slides 30-34 have been included to explain this idea, in case you can’t use the applet.

* Ask the students if we can use this process of drawing dots to create the whole Sierpiński triangle.

If not, why not? Acknowledge that the pen we’re using is thick, and at some point the tiny triangles will be smaller than the dot our pen can draw, so we won’t be able to draw the whole triangle; and even if we use a thinner pen, there will eventually be a point where would stop being able to see the triangles.

## **Sierpiński Triangle construction activity (25 minutes)**

\*GIVE OUT ISOMETRIC PAPER (USE WORKSHEET 07 TO PRINT, OR PRE-PURCHASE)\*

Explain we will now construct a Sierpiński triangle in a different way - by drawing the triangles, starting with a big triangle and adding the middle hole each time.

Ask the students to start by drawing a triangle with 16 spaces along the edge (they’ll need to start at one edge of the paper if it’s going to fit on!), and then to come up with a way to describe the rule that will construct the next triangle they need to draw.

A good rule is ‘find the largest upright triangle and draw in the largest downward-pointing equilateral triangle you can’.

* Ask them why we’re starting with edges of 16 spaces

If they realise each triangle will be at the halfway point of each edge, they’ll need a number that can be halved many times.

Now ask the students to construct as many layers as are shown on slides 35-39, and at each stage to colour in the triangle they’re removing (this will make it easier later to see where triangles need to be removed). If they use a different colour for each size of triangle, they can see how many triangles are added in each layer. There’s a slide 40 including these instructions, and a slide 41 showing the finished triangle only.

If they finish early, they can go beyond what’s on the slide. A slide 42 with an extra layer of triangles is also provided.

Ask some questions about the resulting drawing.

* Have you finished your drawing? If not, why not?
* If you think you haven’t finished because a layer is incomplete, then imagine you have completed that layer. Now is it finished?
* Are there still spaces where you could colour in a triangle?
* When will this stop being true?

The students may find they’re limited by the thickness of the pencil, or the precision of the ruler. Help them realise that each time you draw a triangle, three more blank triangles are created, so you will never stop having triangles to draw.

Explain this is why the Sierpiński is an example of a fractal (show slide 43 with the word ‘fractal’), and that if we zoom in on one it’ll carry on forever. Slide 44 has an animation of the Sierpiński triangle which will loop as long as you need it.

## **Fractal Discussion (15 minutes)**

\*GIVE OUT FILL IN TABLE WORKSHEET 02\*

Further discussion about the interesting properties of fractals can be prompted by filling in the table of the ‘Fractal properties’ worksheet. (NB “Area” is measured in terms of the number of small printed triangles covered). Fill in the first two rows of the table together, and ask the students to fill in the rest. You can use the colours you’ve used to shade each layer to make it clear which triangles are being counted at each stage.

Slide 45 shows the blank table and slide 46 has the table filled in, so you can talk through the answers together. Is there a pattern in the number of triangles (there are three times as many each time - why is this? Each time we draw a triangle pointing down we create three pointing up, each of which have a triangle in them on the next layer).

The size of the triangles gets smaller each time too - if you halve the height of the triangle, what happens to the area?

## **Construct Pascal’s Triangle Activity (10 minutes: can be omitted)**

Explain to the students we’re going to look at a different triangle now - one filled with numbers - but we’ll still be looking for patterns in the triangle. It’s called Pascal’s triangle, and the name is given on slide 47.

\*GIVE OUT BLANK SMALL PASCAL’S TRIANGLE WORKSHEET 03\*

Introduce the blank Pascal’s triangle worksheets, using the image shown on slide 47, and explain the number in each square describes the number of ways to get to that square (starting from the top square), if you obey the rules that you’re only allowed to move down the triangle, but not across or up (these rules are shown on slides 49-51).

There’s only one way to reach the top square (you start there, so you reach it by doing nothing). For each of the squares in the second row, there’s also only one way, but once we get to the third row, there’s a square with two routes (can your students see both the routes?) These numbers are added gradually over the next few slides 52-54.

Slide 55 starts the fourth row. Slides 56-59 demonstrate the three possible ways to get to the second square, while the row is completed in slide 60. Slide 61 gives the numbers for the fifth row.

* Ask the students to describe any patterns they see in the numbers.

They may notice that the numbers are always symmetrical; or they are always 1 at the ends; they get bigger as you go down the side. They may not spot that each number is the sum of the two above (slide 62 highlights a triangle of three cells to see if they can see the pattern if you look at just one pair of cells and the cell below).

* Ask the students to predict what number goes in the cell below 4 and 6 (using the slide 63 that highlights these two cells and the one below). It’s 4 + 6 = 10.

Now give them some time to fill in the rest of the triangle. Slide 64 can be left up with the instructions, and the finished triangle is shown on slide 66 (slide 65has 2 examples, hidden).

**Pascal’s Triangle Activity (and link to fractals) (10 minutes)**

\*IF YOU HAVEN’T CONSTRUCTED PASCAL’S TRIANGLE GIVE OUT WORKSHEET 06\*

Next, explain we’ll use the pattern in numbers to make a pattern of shapes - by colouring in only the even numbers in the triangle. Ask your students to do this and look for patterns in what they get.

They may recognise that this shape (shown on slide 67) is like the Sierpiński triangle - a large triangle in the middle with smaller ones at the sides.

\*GIVE OUT ADDITION RULES WORKSHEET 04\*

To investigate this interesting pattern, ask the students what will happen if you add together two even numbers - will the answer be odd or even? They should be able to fill in the ‘Working out the rules’ worksheet sheet - first, completing the ‘sums’ at the top, then writing and shading in the ‘rules’ boxes below. Ask them to write a simpler way of remembering this rule at the bottom of the sheet (if the numbers are the same parity, colour the cell below and if not, don’t. This is reinforced on a slide as ‘same = shade, different = don’t’). Slides 69-73 are provided showing the blank and filled versions of each part of the worksheet.

\*GIVE OUT LARGE BLANK PASCAL’S TRIANGLE WORKSHEET 05\*

See if the students realise that once you know these rules will work for any odd or even numbers, you don’t need to know the numbers at all - just the colours of the squares above. Provide them with the larger blank triangle (32 rows) and ask them to colour it based on the rules, starting with the three blank cells at the top.

You can also discuss why this set of rules always gives downward pointing triangle shapes - a pair of blank squares will generate a coloured square below, but a pair of squares only coloured on one side makes the square across from it be coloured, creating a slope inwards. A pair of coloured squares gives a coloured square in the middle below, extending the triangle shape.

## **End of session - recap (10 minutes)**

Finish the session using slides 74-77 to remind everyone that you’ve looked at:

* patterns in numbers
* Iteration, using dots
* Fractals like Sierpiński’s triangle
* Pascal’s triangle, and the patterns in the numbers there.

Encourage students when they go out from here, always remember to look for patterns in the world around them, and when they see something interesting, investigate why!

Slide 78is provided to encourage students to send in any questions, and to hand in a post-it note to give feedback. Slide 79 shows some examples of follow-on activities via nrich, which could be used as extension activities or for students to continue their investigations at home.

\*OPTIONAL HANDOUT: Slide 79can be printed out as a takeaway sheet.\*