

Masterclass: Session Script



This icon means there's a slide, or slides in the presentation to accompany this line of the script.



This icon indicates the students will have an activity to do, or something to write.

Introduction (10 minutes)



Welcome to today's Masterclass. To start with, there's an activity for you to work on - can you use these wipeable sheets to draw a single path that crosses all seven bridges, without crossing the same bridge twice? If you get stuck, try starting from a different place.



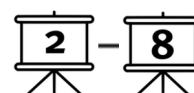
[This next section can be skipped if this masterclass is not the first one in the series.]

These masterclasses are organised by the Royal Institution.

Has anyone heard of the Royal Institution before? It was

founded in 1799, and has always been about letting everyone

have access to science - organising masterclasses and lectures, including the Christmas lectures.



Many famous discoveries have been made in the Faraday building where they are based, including 10 chemical elements. Michael Faraday, who the building is named after, did work there on electricity and optics, which we all use every day.

The Ri are perhaps most famous for Christmas Lectures for young people, which have taken place since 1825. They have been televised for several decades, and many past series are available on the Ri website. The Masterclass programme was born out of the Christmas lectures delivered by Christopher Zeeman in

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1978, which were the very first ones on mathematics!

There are many opportunities for you to visit the Ri building in London, and see the historic rooms for yourself. There is a small museum too. There are lots of talks and holiday events for young people: all the details are on the Ri's website.



The Bridges of Königsberg (15 minutes)

In this Masterclass session, we'll be thinking about drawing paths and looking at how things are connected together. You've been trying to draw a path through this city.

- **Has anyone managed to do this, following all the rules?**

I notice that nobody has their hands up - and this is interesting. As mathematicians, we can see this kind of thing happen and think, maybe there's a reason for it. So we're going to learn some ways to work on problems like these and to understand them better.

This diagram is based on a real city - the city of Königsberg, which used to be in Germany but is now part of Russia (and has changed its name to Kaliningrad).



A mathematician called Leonhard Euler (pronounced 'oiler') used to live in St Petersburg nearby, and someone asked for his help to solve the problem - if you live in this city, can you travel round and visit all seven bridges?



This is a picture of the actual city, but the sheet we've given you has a much simpler version of this picture - to solve the problem we don't need all the extra information about where the houses are, and how many boats are in the river - so this version of the picture just shows the river and bridges.



When Euler worked on this problem, he used an even simpler version of this picture, which looked like this. It doesn't even show the shape of the river, or the layout of the bridges, but just the way the four sections of land are connected.



- **Can you see that these two diagrams represent the same layout? Which section of land does this point on the left represent?**



It's the central piece in the middle of the river, because it has five bridges connecting it to the other pieces of land.

So now, instead of asking the question 'can you find a route that crosses all the bridges', the question can now be 'can you find a route that travels along all these lines', which is the same as saying 'can you draw this shape without taking your pen off the paper and without going over the same line twice?'

The way that Euler found a solution to the problem was to look at other examples of similar problems, to see if he could figure out what makes it possible to draw these shapes. We've got a sheet of examples for you to try here.



For each of these diagrams, try to draw the whole shape with one line without repeating any sections, and mark in the table at the bottom whether you think it can be done. Don't give up too quickly!



This slide shows one possible solution to each of these paths - but the second one is not possible.



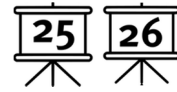
Graphs (10 minutes)

The diagrams we've been tracing here are examples of what mathematicians call graphs - not the kind of graphs you might have studied before, with axes and a plot or bars; but graphs which are networks of points. Here's some new words you'll need to know about graphs:

- **Vertex:** this is the word we use to describe a point - one of the dots that makes up the graph and is connected to the other dots
- **Edge:** this is the word we'll use to mean one of the lines connecting two points.



- Graph: this is the word for the whole shape - it has to be connected in one piece, and every edge has to start and finish at a vertex
- Euler path: this is what we've been trying to find - it's a path that starts somewhere in the graph and goes along each edge exactly once. If you can find an Euler path, that's the same as saying you can draw the graph without taking your pen off or repeating a line.



Here are some examples of things which are NOT proper graphs.

- **Can anyone tell me why, and how we can fix each of them?**

Let's start with this one in the top right. It's got an edge that crosses here without a vertex at that point - which means all these edges meeting in the middle don't have a vertex at both ends. How could we fix this? We could add a vertex here - or, we could move this vertex and edge down to stop it from crossing. **Is it ok now?**




What about this one in the bottom left? It's missing some vertices. We can't fix it by moving the edges, but we could add two vertices.



This one has an extra edge here with no vertex at the end - so we could fix it by adding a vertex at the end, or removing that edge.



 Now I'd like you to all design your own graph - remember all these rules - you can't have the end of an edge, or two edges crossing, without a vertex there, and it has to be one connected piece. When you've finished, write down how many edges and vertices you have in your graph.




Odd and Even Vertices

One way that Euler used to understand graphs better was to look at each vertex, and count how many edges meet there. For example, in these examples of graphs we've been looking at, these points have one edge, these have two, and these have three.



Euler also thought it was important to know whether the number of edges meeting at a vertex is **odd** or **even**. I'm going to use a different colour to mark which of these are odd numbers and which are even numbers.



 For each graph, count how many edges meet at each corner and label them in two different colours for odd and even, and fill in the table at the bottom. These are the same graphs from the previous sheet, so we know



whether they're possible or not. Copy that information across into this table too. See if you can find a pattern!

If you've filled in the whole table it should look like this.



- **Can anyone see a pattern in this table?**

If your graph has two odd vertices, it can be done - and if it has zero odd vertices, it can also be done. But if it has four odd vertices, it can't be done.

This isn't just a pattern - it's always true for any graph that if it has 2 or 0 odd vertices it has an Euler path - it can be drawn - and if it hasn't, then it can't.



Take some time to think about why this might be true, and discuss it with people nearby - look at the paths you've managed to draw on the ones which are possible, and where the vertices are. Think about what happens when you pass through a vertex.

You might have noticed that when you pass through a vertex, you go in along one edge and out along another, so that uses two of the edges from that point. That means if there's an odd number of edges, you can use up all but one of them by going in and out. The only way to use up the remaining edge is if you start or finish at that vertex, then that edge is the first or last edge that gets used.

This means if there are exactly two odd vertices, these will be the start and finish of your Euler path.

- **If you didn't already notice this, check your paths now and see that you always start and finish at the odd vertices.**

If all the vertices are even, you can still draw an Euler path, but it doesn't matter where you start and finish - you can even start and finish in the same place!

Now we can look back at the Bridges of Königsberg graph, and see that this has four odd vertices - which means it's not possible. If you couldn't do it, you shouldn't feel bad, because it can't be done! In fact, by understanding the reasons why it can't be done, we've solved the problem. This was the answer that Euler came to after all his work - but at the same time he'd also managed to work out this way of understanding graphs, and he hadn't just solved the problem for this graph - but for all graphs at once!



Euler's Formula (15 minutes)

One other way Euler used to study graphs was to count how many edges and vertices they have - and also how many faces. When I say faces, I mean regions - areas of space closed off by the edges, including the one region outside the whole graph. So here are some examples of the numbers of vertices, edges and faces for each of these graphs.



Euler came up with a formula which connects these three numbers, and it was to take the number of vertices - V , minus the number of edges E , plus the number of faces F . So we're going to work this out for all the graphs on our sheet, and some new ones.



Use this sheet [Euler Formula Worksheet] to write down V , E and F , and $V-E+F$ for all the graphs on the first sheet (which I'll also put on the screen) and for these four new ones.



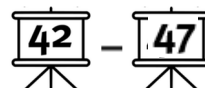
Now you've filled in this table, you should find a pattern in the numbers.

- **Can anyone tell me what the pattern is?**



It looks like $V - E + F$ is always equal to 2! This is the pattern Euler found, and it will always work as long as your graph is correct.

To understand why, we can start with a very simple graph. If we just have one vertex, V is 1 and F (the outside region) is 1.



What's $V - E + F$? $1 - 0 + 1 = 2$.

Now we can add one edge - and there are a few ways to do this. We could add an edge from this vertex to itself. If we do this, we have 1 edge now but also 1 extra face, as we've closed a region. **So what's $V - E + F$?** $1 - 1 + 2 = 2$.

So if instead of adding an edge that joins back to the same vertex, we add one that doesn't, we'll need an extra vertex on the end of it. We could do this - **now what's $V - E + F$?** $2 - 1 + 1 = 2$.

If I want to just add an extra vertex, I could do that by adding one here in the middle of an edge. But now instead of one edge, this is two. **So what's $V - E + F$?** $3 - 2 + 1 = 2$.

And finally, if I want to add a face - a closed region - I could do that by adding an edge like this that goes from this vertex to this one, and closes off this face. **What's $V - E + F$ now?** $3 - 3 + 2 = 2$! So this will always work.

You might be thinking that you already know the words vertex, edge and face from somewhere else.

- **Can anyone suggest what else has vertices, edges and faces?**

You may have heard these words when talking about 3D shapes, like this cube or this prism. And this formula of Euler's also works for simple shapes like these!



- **How many edges does a cube have? How many vertices? And how many faces? And what's $V - E + F$? And what about for the prism?**

City of Königsblank (10 minutes)

One final thing before we finish. Now we understand how to make a graph which DOES have an Euler path, I want you to redesign this city. The islands will be the same shape, but you can draw in your own bridges. I want you to do this so that this city does have an Euler path in it - so you can cross all the bridges in one journey, and not cross any bridge twice. You do need to be able to get to all the parts of the city!

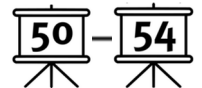


You can also give your made-up city a name, and draw in some other famous made-up landmarks that someone who follows the Euler path might visit.

Remember what the rules are for something to have an Euler path? **Only two or zero odd vertices!** If it helps, you can draw the graph first on another piece of paper, and then put in the bridges on the map.

End of session - recap

In this session we've looked at a famous old puzzle called the **Bridges of Königsberg** puzzle, and decided it can't be solved - but not because we gave up! It was because we understand how to solve this kind of puzzle now, and we know for sure it can't be done. We also learned about **graphs**, and what an **Euler path** is. And we also know **Euler's formula**, which works for graphs like these, and for simple shapes in 3D as well.



Now you can challenge your friends to draw shapes like these, and if they get stuck you could tell them to count the edges at each vertex. And if you find any new 3D shapes, check if they also work with Euler's formula, by counting the vertices, edges and faces!