

Inspiration:

This workshop introduces the area of mathematics known as graph theory, and how it was discovered. A graph is a diagram consisting of vertices (points) and edges (lines). The idea of a graph is to show how objects are connected without worrying about actual distances, angles or the geometric shape of the graph.

We'll discuss the historical 'Bridges of Königsberg' problem, and explore ways to study graphs in order to categorise them - into those which contain an Euler path (a single route passing along each edge once without repeats) and those which do not, and find a rule for which is which. We'll also consider Euler's formula which he discovered applies to all valid graphs, and how it applies to other shapes. We'll finish by redesigning the city of Königsberg to make the problem solvable.

Overview of Activities:

- Bridges of Königsberg activity
- Path tracing activity
- Odd and even vertices activity
- Euler Formula activity
- The city of Königsblank

General Masterclass resources needed:

- Register of children
- Consent forms and emergency information to hand
- Stickers and markers for name badges
- Adult register
- Ri child protection policy
- Paper and pencils/whiteboards for workings
- 2 different coloured post-it note pads
- Drinks and biscuits

Specific resources needed (printable worksheets in worksheet folder):

- Bridges of Königsberg laminated worksheet (printed A4 or A5) per student, dry wipe marker
- Path tracing worksheet per student
- Odd and even vertices worksheet per student
- Coloured pens or pencils (two colours per student)
- Euler formula worksheet per student
- Map of Königsblank per student

Support resources:

- Helper notes: An overview of the Masterclass content and activities
- Supporting notes: Extra information and background on the content of this Masterclass
- Session script: Suggested wording for each section of the session

Things to prepare in advance

- Print worksheets and resources as detailed above
- Gather general Masterclass resources

Ask the Ri

Don't forget to collect any questions which arise, and email them to the Masterclass team at the Royal Institution: masterclasses@ri.ac.uk

Feedback

We would very much welcome your feedback on this session. If you have time, please collect feedback from the students at the end of the Masterclass and send it through to us. We would also appreciate

feedback on how you have used the session, what you think worked well and what improvements would be useful.

Time plan of Masterclass:

Slides & Time	Overview	Activity
Slide 1 5 minutes (5)	Introduction Instructions on screen. Helper and Speaker circulating and chatting with students	Settling activity - Bridges of Königsberg Instructions on a slide - give students laminated ' Bridges of Königsberg ' worksheets and dry wipe markers to try to complete the task. They must try to draw a single route that goes through the city and crosses all the bridges, without crossing over any bridge twice. They can try starting from different places if they get stuck. This task is not possible, so if anyone thinks they have solved it, they may have misunderstood the question - guide them to try again making sure they understand the rules.
Slides 2-8 5 minutes (10)	Introduction to the Ri <i>[Only include these slides for the first session in the series – otherwise remember to hide the slides before you start the Masterclass]</i>	Use these slides to introduce the students to the work of the Ri and other ways they can get involved – see notes on the slides for more detail. In particular: <ul style="list-style-type: none"> • The Ri is a science communication charity which has been around since 1799. We've got a huge amount of history and lots of famous scientists lived and worked at the Ri. Most importantly, we've always been about communicating science to the general public – and that's something we still do today. We do talks and activities for the public as well as with schools all across the UK. • There are lots of family events at our building in London, including family fun days and holiday workshops just like the Masterclasses. • The CHRISTMAS LECTURES are for young people and are on television at Christmas time, looking at a different topic every year. We've got an archive on our website of all of the recent series plus many of the older ones. The CHRISTMAS LECTURES are what started the Masterclass programme. See slide notes for links. • We have a YouTube channel with lots of videos for people interested in science (and maths engineering, computer science...), especially our ExpeRimental series which is all about doing experiments at home. • Students are part of a big family of Masterclass attendees – we have been running Masterclasses since 1981. • Students at series running within reach of London will be invited to a Celebration Event at the Ri in June/July. • You can become an Ri Member to get more involved with what we do (and enter the ballot to buy tickets to the CHRISTMAS LECTURES filming).
Slides 9-15 5 minutes (15)	Discussion of introductory task Explanation of Königsberg historical problem	Ask students if anyone managed to solve the problem. If anyone has solved it, check their answer (it is not possible, so they will have done something wrong or misunderstood the question). Discuss that it's interesting nobody solved it - sometimes in maths if you see something like this - that looks like a pattern - it might mean something, and it's worth investigating. Describe the historical problem of the Bridges of Königsberg in context - Leonhard Euler studied it in the 18th century, and it inspired him to study some interesting maths. The city has four areas of land, and seven bridges connecting them. Explain that Euler tried to study this by simplifying the diagram and reducing each area of land to a single point, with lines between if they are connected by a bridge. Can your students see that the two diagrams are equivalent? Explain that now instead of trying to

		<p>cross all the bridges, we have to be able to draw the whole shape without going over any bit of line twice, and without taking our pen off the paper.</p> <p>Euler studied this problem by looking at other possible shapes you might try to trace - so we'll investigate the same way.</p>
<p>Slides 16-17</p> <p>10 minutes (25)</p>	<p>Path tracing activity</p> <p>Helper and Speaker circulating and chatting with students</p>	<p>Give students the 'path tracing' activity sheet, and see if they can draw each of the diagrams without taking their pen off and without going over any section of line more than once. Ask them to fill in the table at the bottom of the sheet when they've decided if it can be done or not - and don't let them give up too soon!</p> <p>One solution for each path (that is possible) is provided on a slide to show the students. They may have found a different way!</p>
<p>Slides 18-31</p> <p>10 minutes (35)</p>	<p>Graphs and useful words</p> <p>Explanation of terminology</p> <p>Helper and Speaker circulating and chatting with students for graph design activity</p>	<p>Explain to students that the diagrams they've been drawing are sometimes called graphs (not to be confused with graphs for showing data - these are instead networks of points, but mathematicians use the word graph for them). Define the following terms:</p> <ul style="list-style-type: none"> ● Vertex: one point in the graph - a dot to join to others ● Edge: a line between two vertices ● Graph: a diagram made up of edges and vertices, where every edge has a vertex at both ends ● Euler Path: a single path running through the graph, going along each edge exactly once <p>Show some examples of non-valid graphs, and ask students how each graph could be repaired by adding vertices or removing edges.</p> <p>Then, ask the students to design their own graph on a blank piece of paper. Ask them to write down how many edges and how many vertices it has, and to make sure every edge has a vertex at both ends (and if two lines cross each other anywhere, they must draw a vertex there).</p>
<p>Slides 32-38</p> <p>25 minutes (60)</p>	<p>Odd and Even vertices</p> <p>Explanation of terminology</p> <p>Helper and Speaker circulating and chatting with students</p>	<p>Introduce the idea of the degree of a vertex - the number of lines which meet there (one which goes through and out counts as two lines). Ask the students to work out the degree of the vertices in all the graphs on the sheet, and the one they've drawn.</p> <p>Then introduce the concept of odd and even vertices - if it has an odd number of lines meeting, it's odd and if it has an even number, it's even. Ask the students to work out which vertices in their graphs are odd and even - they could use two different colours to circle the two types.</p> <p>Next ask them to fill in the table on the 'Odd and even vertices' worksheet, and see if they can see anything interesting. Do they see a pattern in the table? They should find that graphs which have an Euler path have either 2, or 0, odd vertices.</p> <p>Can the students explain why this is? Ask them to think about what is involved in drawing a path - what happens at the start and finish? What happens when they pass through a vertex in the middle of the path?</p> <p>When you pass through, you go in along one edge and out along another, so that uses two of the edges from that point. Unless</p>

		<p>you start and finish in the same place, you'll need an extra edge on the two end vertices, to go into or come out from them. So if they are all even, you start and finish in the same place, and if two are odd, you can start at one and finish at the other.</p> <p>Now we can look back at the Bridges of Königsberg, and see that this has four odd vertices - which means it's not possible. Emphasise that having found a problem difficult doesn't mean they are bad at maths - sometimes the answer is that it's not possible, and by proving it's not possible and understanding why, they have succeeded in solving this problem.</p>
10 mins (70)	BREAK	Drinks and biscuits and comfort break
Slides 39-49 15 minutes (85)	<p>Euler Formula</p> <p>Helper and Speaker circulating and chatting with students</p> <p>Explanation of why this happens</p>	<p>Explain that another thing Euler studied about graphs was to look at what is called the Euler formula of a graph - it's calculated by taking the number of vertices, edges and regions (including the outside of the graph) and adding them using the formula $V + E - F$.</p> <p>Ask students to calculate the value of the Euler formula for all the graphs on the sheet and fill in the table on the 'Euler Formula' worksheet. They should find the answer is always 2. Ask them to design their own graphs, following all the rules, and see if they always get 2.</p> <p>To understand why this happens, we can start with a simple graph consisting only of one vertex. How many edges? None. How many vertices? 1. How many regions? 1. $1 - 0 + 1 = 2$. Now look what happens if we add an edge. If the edge connects to itself, that creates an extra region too so the formula still holds. If it doesn't, we'll need another vertex on the end, but no new regions are made so the formula still holds. This will always carry on every time we add an edge, or divide an edge in two by adding a vertex.</p> <p>Ask if the students have heard the words 'vertex' and 'edge' before - they may have called vertices 'corners', when talking about 3D shapes. Explain that this formula also works for some 3D shapes - show the slide with a cube, and a triangular prism and ask the students to count the vertices, edges and faces and calculate the Euler formula.</p>
Slide 50 10 minutes (95)	<p>Redesigning Königsberg</p> <p>Helper and Speaker circulating and chatting with students</p>	<p>Now that we understand why the city of Königsberg wasn't possible to traverse using the bridges once each, give the students a copy of the map of the city of Königsblank, which has all the same islands but no bridges.</p> <p>Ask them to place some bridges so that the city is connected up but it does have an Euler path - they can name their made-up city, draw in some famous landmarks and mark in an Euler path that visits them, by crossing all the bridges once. Before they start, check that the students can describe the properties their city's bridges must have.</p>
Slides 51-54 5 minutes (100)	<p>Feedback, tidy up, questions time</p> <p>Ask the Ri</p>	<p>Recap of session contents.</p> <p>Don't forget to collect any questions and feedback on post-it notes, and email them to the Masterclass team at the Royal Institution: masterclasses@ri.ac.uk</p>
Slides 55-58	Possible NRICH problems related to this session – use as extension activities or for them to do at home	<p>nrich.maths.org/2326 - Tourism</p> <p>nrich.maths.org/2327 - Königsberg plus</p> <p>nrich.maths.org/11821 - Who's who?</p> <p>nrich.maths.org/957 - Rail Network</p>