Bridges of Konigsberg Masterclass

Thanks for helping with this Masterclass session! Your support is much appreciated.

The session leader should be able to tell you more about the content of the session, and exactly how they'd like you to help, but this sheet should give you some basic information you may find useful. If any of this seems obvious to you, that's great!

In general, for Masterclass sessions:

- While the session leader is talking to the group, don't interrupt them or distract the students unless something is wrong that needs fixing urgently. You should also watch and pay attention to what they're saying, to set a good example.
- If things need handing out to the students, wait for the session leader to signal you to do this, as it can distract the students if you start to hand things out before they're ready.
- If the students are given a task to work on, you should circulate the room to talk to the students. Wait until they've had a chance to tackle the problem before you interrupt them, and encourage anyone who looks like they haven't started yet.
- Try not to give away the answers to the students, especially if they're working on the problem and about to discover it for themselves - if they are really struggling, you can give them a hint or suggest where they might start looking.

In this session:

This workshop is an exploration of graph theory, starting from the famous Bridges of Königsberg problem. Students will have the chance to learn some of the ideas and terminology from graph theory, and how they can apply it to simple path tracing problems.

The main activities are:

1. **Bridges of Königsberg activity**
2. **Path tracing activity**
3. **Odd and even vertices activity**
4. **Euler Formula activity**
5. **The city of Königsblank**

There is a great deal more background information available on a separate sheet, if you would like more detail. Please ask the session leader if you'd like to see it.

Thanks again for your help with this session! If you have any other questions, please ask the session leader.

In this session:

We'll be working with graphs and networks, and will use the following terminology:

- Vertex (for a point, or node within a graph)
- Edge (for a line, or arc within a graph)
- Face (for a region within a graph, including the outside region)

If you can make sure you try to use these consistent names for these objects that will align with the wording on the worksheets and slides.

Bridges of Königsberg activity

Before the session introduction, students will be given the task of attempting the 'Bridges of Königsberg' puzzle, using laminated worksheets and dry wipe markers. They must try to draw a single route that goes through the city and crosses all the bridges, without crossing over any bridge twice. Note that this is different to trying to visit all the islands - they need to cross every bridge exactly once, in one direction or the other. If they get stuck, suggest they try starting from a different place, if they haven't tried that already.

This task is not possible, so if anyone thinks they have solved it, they may have misunderstood the question - guide them to try again making sure they understand the rules. You should also take care not to let students get too frustrated by the task - they won't be left to work on it for very long, but if you see anyone who becomes frustrated with it and convinced it's not possible (or who has seen it before and knows it's not possible) you could ask them to think about why they think it's not possible - and to try to work out what makes it impossible.

Path tracing activity

Students will be asked to complete the 'path tracing' worksheet, which involves trying to draw a single path on each diagram which traces over every line once and doesn't go over any line, or part of a line, more than once. They may do this on the sheet, or have a separate sheet of paper where they attempt to redraw the shapes.

Students should have had it explained that tracing the 'City of Konigsberg' graph is equivalent to trying to solve the puzzle with the bridges - again, it's not a case of visiting all the vertices, but of travelling along every edge exactly once in either direction (to do this, they may visit the same vertex twice, as long as they arrive/leave along arcs they haven't used already).

They've been asked to fill in a table at the bottom of the sheet to determine whether the shape is possible to trace or not, although they shouldn't give up too soon - most of them can be done. The solutions to this sheet are available from the session leader, so you can see which ones can't be done, and an example of how to do it (these can be shown on a slide after the activity).

The next activity will be discussing odd and even vertices in the graphs (ones with an odd, or even, number of lines meeting there) and in general the graphs with exactly two odd vertices will use a path which starts from one of the odd vertices and ends at the other. Don't worry about explaining this to the students, as they will be about to do this all together, but if students are really stuck, you could suggest they start from one of those two vertices, or at least see which vertices they've tried starting from already and suggest they try another. If the graph has no odd vertices, it can be traced starting from anywhere (and a path finishing at the same point it started can be found).

At the end of this task, the definition of a graph will be given, and students will be asked to draw their own example of a graph. They will see some examples of non-valid graphs first, but you'll need to make sure their graphs satisfy all the criteria:

- the graph is a single piece, all connected together
- every edge has a vertex at both ends (this can be the same vertex - a loop can come back to where it started and this counts as two lines meeting there)
- if two lines cross each other anywhere on the graph, they must meet at a vertex there - you can't have any crossing or junction without a vertex.

The session leader will also ask them to write down how many edges and how many vertices their graph has, and if they miss any you can help by counting.

Odd and even vertices activity

The students will be given a second sheet with the same seven diagrams, and asked to write how many lines meet at each vertex, and then to class each vertex as odd or even. They can use two colours to circle the odd and even vertices, and then fill in the table to say how many of each are present in each graph.

They can copy over the information about which graphs are traceable, to see if they can find a pattern in the data - encourage them to do this. The pattern they're looking for is as described above - any graph which can be traced will either have exactly two odd vertices, or no odd vertices.

Euler Formula activity

Next we'll analyse a little more about the structure of graphs by counting the vertices, edges and faces. In this case, faces are any enclosed region within the graph entirely closed off by edges, and includes the outside region which counts as one face.

The students will calculate the Euler formula (V - E + F) for each of the graphs on the sheet. The first seven examples are the same ones seen previously, and then a further four examples on the worksheet with the table - students will need to refer to the previous worksheet to see the other 7 graphs, or they can be displayed on a slide during the activity. They can also do this for their own graph they designed.

The students should find that V - E + F = 2, for all graphs. If this doesn't seem to be the case, check they've counted all the vertices, edges and faces (forgetting the outside region might happen a lot!) and that they've done the calculation correctly. If it's their own graph, check it's a valid graph and there are no joins between edges missing vertices.

The city of Königsblank

The final activity in the session will allow students to design their own set of bridges in a fictional city with the same land layout as Königsberg, so that the city can be traversed. For this to work, they should have either none or exactly two sections of land reachable by an odd number of bridges. It should also be possible to reach any part of the city using the bridges.

If it helps them to plan, you could show them how to draw the vertices only from the equivalent graph (four corners of a horizontal diamond), design their city there and transfer it to the map. They can be encouraged to design and name their new made-up city, and draw in some fictional maths landmarks. They can also draw in an Euler path (one which crosses all the bridges) through the city.