Bridges of Königsberg Masterclass: Extra Background Information

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The city of Königsberg was founded in 1255 in what was then part of Germany, called Prussia. Since 1945 the city is part of Russia and has been renamed Kaliningrad. Historically, the city was laid out across a fork in the River Pregel, with a section of the city on an island in the middle of the river.

Image: Historic Cities Research Project

People who lived in the city wondered over coffee whether it would be possible to cross all of the city's bridges in one journey, without crossing any of the bridges twice. Each bridge would be crossed exactly once in some direction. People struggled to find a solution to this problem, and it took a mathematician - Leonhard Euler, who lived in nearby St Petersburg - to find the answer.

Euler was a Swiss mathematician, physicist, astronomer, logician and engineer, born in 1707 in Basel, Switzerland. He was well known for making many great and deep contributions to mathematics, and has more published works than any other mathematician ever. He also introduced much of the modern mathematical terminology and notation that is still used today.

Among his many contributions was the theory of graphs, which was inspired in part by his work on the Bridges of Königsberg problem - the paper he published on the problem in 1736 is regarded as the first in the history of graph theory.

Euler’s solution was to prove that a complete path through the city, crossing all bridges once, was not possible. He discovered this by first working out a general rule for all similar problems, then applying this understanding to the specific bridges problem. Often in mathematics, generalising a problem to see how it works in different cases can be a great way to get an understanding of the overall problem.

Graph Theory

Graphs are networks of points (**vertices**; singular: vertex) connected by lines (**edges**), and can be used to model all kinds of real-world and theoretical problems. The idea of a graph is to show how objects are connected without worrying about actual distances, angles or the geometric shape of the graph. Sometimes edges and vertices are called **arcs** and **nodes**.

You could draw a vertex for each of the numbers 1-10 and connect them with an edge if the numbers share a common factor; you could model the friendships within a group of people using a graph; or you can use them to describe physical networks - graphs are useful in designing, modelling and planning networks such as train lines, utilities supply, delivery routes and many others.

In these kinds of real-world networks, there’s other important information, like the distances between stops, or how long it takes to travel - but you don’t need this information to draw a graph. Many train route maps are examples of a graph, as the actual distances between stations aren’t represented, rather just the connections between the stops.


This difference can be seen in these two copies of the London Tube Map - on the left (image from [TFL.gov.uk]) the map is laid out to clearly show the connections between the stations, but the positions of the stations are not the same as they are on a real surface map. On the right (image from [london-tubemap.com]) the positions of the stations and lines are accurate, but this map is less useful to plan a journey - some stations are bunched up close together, and there are large gaps which make the map bigger than it needs to be.

It is possible to store more information in a graph than just the way things are connected - some graphs have arrows connecting the vertices instead of just plain edges, and these are called **directed graphs** or **digraphs** - they store information about which way you travel along the edge, if this information is important. It's also possible to assign a number (a weight) to each edge, so you could encode distances or travel times, and this is called a **weighted graph**. In this workshop, we deal only with simple unlabelled graphs.

One property of a graph discussed in this workshop is the number of odd and even vertices in the graph - a vertex is odd or even depending on whether it has an odd or even number of edges meeting there. The number of edges meeting at a vertex is sometimes called the **degree** of the vertex.

Planarity

All of the graphs we deal with in this workshop are **planar graphs** - this means they can be drawn on a flat piece of paper without any of the lines crossing. If you want to draw some more complicated graphs, it's not possible to do this without any crossings - one nice example is called K5, the complete graph on 5 vertices. Draw 5 vertices in a pentagon shape, and try to connect every vertex to all four of the other vertices. You should be able to draw some of the edges in, but you'll reach a point where it's not possible to add any more without crossing an existing line - and it's been proven that it's not possible to rearrange the graph, or any of the edges you've drawn already, to do this.

We've defined a graph as having a vertex at every point where two lines meet or cross - but if you were drawing a non-planar graph like K5 you would need to have a crossing without a vertex, to represent where the two edges pass over and under. To keep things simple, we deal only with planar graphs in this workshop.

Euler Paths

The problem of 'crossing all the bridges' in Konigsberg can be seen as equivalent to the following problem: if the city is drawn as a graph with a vertex for each part of the city and edges where the bridges connect the different parts, can we find a path which travels along each edge exactly once? While there are many properties a graph can have, this one in particular relates to Euler Paths.

We define an Euler Path as one which travels along all the edges, in some direction, exactly once each. It may involve visiting a vertex more than once, as long as that vertex has enough edges connected there. (The problem of traversing a graph and visiting all the vertices once is a separate question.) It's also possible, for some graphs, to create an Euler Circuit - one which travels along all the edges and finishes at the same vertex it started at.

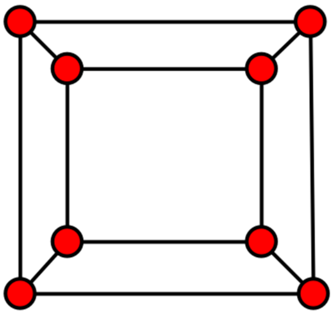
In the workshop, we discover that if a graph has exactly two odd vertices, it's possible to find an Euler Path starting at one of these two vertices and finishing at the other. If the graph has no odd vertices at all, it's possible to create an Euler Circuit, starting and finishing at any vertex. We call a graph with no odd vertices an **Eulerian graph**, and one with exactly two is called **Semi-Eulerian**.

It is amusing to note that as of now, due to historical restructuring after bombing damage in the war and other building since, two of the original seven bridges are no longer standing - so it is now possible to cross all the bridges of Konigsberg (Kaliningrad) in one Euler path!

Euler formula

The formula given in the workshop that relates edges, vertices and faces (regions) in a graph is called the **Euler Formula** - it's sometimes also called the **Euler Number**, or **Euler Characteristic**, denoted by the greek letter χ (chi). A face in this case is defined as any region enclosed by edges, and the outside region of the graph also counts as one face.

In the case of graphs drawn on a flat page, the formula $V - E + F$ will always equal 2. Importantly, this only works for planar graphs - graphs where none of the lines cross each other not at a vertex.



The formula also works for simple 3D shapes like cubes and prisms - if you count the number of edges, vertices and faces they will add up in this same way. We also use the terms edge, vertex and face when describing 3D shapes, and this connection to graphs is important - for example, you can draw a graph showing how the vertices of a cube are connected by stretching one face of the cube to be larger, and looking at it from the end.

Image from Wikimedia user BetacommandBot CC BY-SA

The formula breaks down for more complex shapes - a shape with a hole through the middle will have $V - E + F = 0$, and in general it will equal $2 - 2g$, where g is the number of holes through the shape. This is a very useful tool in studying 3D shapes, and the Euler Formula is very important in **topology**, which is the study of shapes and higher dimensions.

Since Euler was so prolific, there are other mathematical formulae called Euler's Formula which aren't related to this - so make sure you get the right one!