Numbers of Nature Masterclass: Session Script

The Royal Institution
Science Lives Here
Masterclass network

This icon means there's a slide, or slides in the presentation to accompany this line of the script.

This icon indicates the students will have an activity to do, or something to write.

This icon means there is an accompanying Geogebra applet, listed in the resources for this session, which you can load in a browser.

Introduction (10 minutes)

Welcome to today's Masterclass. To start with, there's an activity for you to work on - can you complete these sequences by working out what comes next, and give a rule for how to find the next number?

[This next section can be skipped if this masterclass is not the first one in the series.]

These masterclasses are organised by the Royal Institution. Has anyone heard of the Royal Institution before? It was founded in 1799, and has always been about letting everyone have access to science - organising masterclasses and lectures, including the Christmas lectures.

Many famous discoveries have been made in the Faraday building where they are based, including 10 chemical elements. Michael Faraday, who the building is named after, did work there on electricity and optics, which we all use every day.

The Ri are perhaps most famous for Christmas Lectures for young people, which have taken place since 1825. They have been televised for several decades, and many past series are available on the Ri website. The Masterclass programme was born out of the Christmas lectures delivered by Christopher Zeeman in

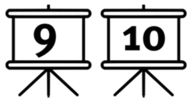
1978, which were the very first ones on mathematics!

There are many opportunities for you to visit the Ri building in London, and see the historic rooms for yourself. There is a small museum too. There are lots of talks and holiday events for young people: all the details are on the Ri's website.

Sequences and rules (25 minutes)

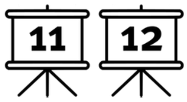
You've had some time to look at the sequences on the first slide - now we can look at each one together and share our answers.

- **1, ½, ⅓, ¼, ⅕, ...**



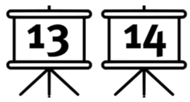
This is a sequence of fractions, and they're getting smaller. The number on the top is always 1, but the number on the bottom increases by 1 each time. The next number would be ⅙.

- **1, 4, 9, 16, 25, ...**



This sequence contains square numbers - the first term is 1×1=1, then 2×2=4, then 3×3=9 and so on. The next number would be 36. To find the 10th number, we could calculate 10×10=100.

- **2, 2, 2, 2, 2, ...**



This sequence is just the number two repeated. The rule for finding the next number is that the number is always two. Even though this seems silly, this is still a sequence of numbers!

Once you have a rule that tells you what the next number in the sequence is, you can keep applying the rule as many times as you want, and the sequence will keep going. Different rules will give sequences that behave differently when you do this - for example, what will happen to this first sequence? It will get smaller. What will happen to the second sequence? It will get bigger. What about the third one? Yes, it will stay the same - it's always 2.

One interesting kind of sequence is one which has a **limit**. We say that a sequence has a limit if it keeps getting closer and closer to a particular number. It might do this by getting smaller, or by getting bigger, or by getting bigger then smaller, as long as it gets closer and closer to a single value. The sequence may never reach the limit, but it never gets further away from it.

For these three sequences that we've just looked at, some of them have a limit and some don't.

Discuss with your neighbour for 30 seconds - which ones have a limit, and what is the limit if there is one?

- **1, ½, ⅓, ¼, ⅕, ...**

The numbers are getting closer and closer to zero - so the limit of this sequence is 0.

- **1, 4, 9, 16, 25, ...**

These numbers keep getting bigger and bigger - they're not getting closer to a particular number, so this sequence does not have a limit. Does anyone think it does have a limit? You might be tempted to say that it has a limit of infinity - because the numbers are getting closer and closer to infinity. But the limit of a sequence has to be a number, and infinity is not a number!

- **2, 2, 2, 2, 2, ...**

This sequence does have a limit. The limit is 2. Even though the numbers aren't changing, so it's not getting closer, the limit is still 2 - the next number is never further away from 2 than the previous one.

Next we're going to look at a worksheet with some more examples of sequences. What's the rule for each sequence? And does that sequence have a limit? Work on the sheet.

- **1, ½, ¼, ⅛** followed by 1/16, 1/32, 1/64

This sequence is getting smaller, but differently to the one we saw previously - the number on the bottom gets twice as big each time, which is the same as saying that we divide the previous number by 2. The sequence will keep getting closer and closer to zero, so the limit is 0.

- **1, -½, ⅓, -¼** followed by ⅕, -⅙, 1/7

This sequence is called an alternating sequence - it changes between a positive number and a negative number each time. As well as this, the fractions are getting smaller each time - the number on the bottom of the fraction increases by 1. This sequence has a limit, and it's zero - even though some of the numbers are above zero and some are below, and even though the numbers aren't always getting bigger or smaller (they get bigger, then smaller, then bigger again) the distance between the number and zero is always getting smaller.

- **0.3, 0.33, 0.333, 0.333** followed by 0.3333, 0.33333, 0.333333

This sequence can be described as 'put a three on the end of the number each time' - but what does this mean? We add three tenths, then three hundredths, then three thousandths, and so on. If we continue forever we'd have 0.3333 with an infinite string of 3s afterwards. Does anyone know what number that makes? ⅓. This sequence is always getting closer to ⅓, and so the limit of the sequence is ⅓. 23

- **5, 5, 5, 5** followed by 5, 5, 5

This is another silly sequence, like the sequence of 2s - and the limit of this sequence is 5, as before. 24

- **1, 3, 6, 10** followed by 15, 21, 28

This sequence is called the Triangular Numbers, as each number can be drawn as a set of dots in a triangle. Each number is the sum of the numbers from 1 up to the number - the 5th number will be the sum of the numbers from 1 to 5, which is 1+2+3+4+5=15. This means you can get from one number to the next by adding the next number - from 15 to 21 you'd add 6. Because these numbers get further apart each time, they're not getting closer to a limit, so this sequence has no limit. 25

Next we're going to play a game - it's called, **"How Far Do You Have to Go?"** To start with, you can challenge me, and I'll try to answer - then you can play against each other. 26

The way the game works is, we start with a sequence. Importantly, the sequence has to be one which has a limit. Let's use this example, from the start of the session - it has a limit, and the limit is zero. 27

- **1, ½, ⅓, ¼, ⅕, ...**

The way you challenge me is to pick a small number - as small as you like. I have to then work out how far along this sequence you'd need to go until I'm closer to the limit than the number you've challenged me with. So, if you were to pick 2, I'd win automatically, because all the numbers in the sequence are less than 2 away from the limit, which is 0. So you would need to pick something smaller - maybe, ⅔? 28

I would then have to find how far I'd need to go along the sequence before I get a number that's closer to 0 than ⅔ is - in this case, the answer is ½. Sometimes it's difficult to tell with fractions which number 29 is smaller - in which case, you can use your calculator to work it out as a decimal, and see which one is smaller.

- **Would anyone else like to challenge me with this sequence?**

Now get into small groups and challenge each other. You'll need to choose a sequence to use, and you can pick one of the sequences on the sheet, or one of the other ones we've talked about today, or another one if you prefer!

Have you enjoyed playing this game? This is one way that mathematicians test whether or not a sequence has a limit - if you can always answer a challenge in this game, no matter how small a number you pick, then the sequence has a limit.

Fibonacci Limits (20 minutes)

Next we're going to look at one sequence of numbers in particular, and that's called the **Fibonacci Sequence**. Hands up if you've heard of it before? Can anyone tell me how the sequence is defined?

We create the Fibonacci sequence using this rule: Each term is the sum of the two previous terms. This doesn't make sense unless you already know how the start of the sequence goes, so we always start with a 1 and another 1, since that's a nice simple way to start, So if the first term is 1, and the second term is 1, what's the third term? It's 2.

Write down the first **ten terms** of the Fibonacci sequence using this rule.

Here are the first ten terms of the Fibonacci sequence.

- **Does this sequence have a limit?**
- **What is the limit of this sequence?**

The sequence will keep getting bigger and bigger, as we're adding together numbers each time; this means it has no limit.

If we want a more interesting sequence, we can study the way this sequence gets bigger - if we take each number in the sequence, and divide it by the previous one, it will give us a new sequence of numbers. These are called **ratios**.

Use this worksheet to work out the ratios between each pair of terms in the Fibonacci sequence. Use a calculator to find the value of the fraction if it's not a whole number.

If you've filled in the whole sheet you should have this sequence of numbers. They're not all whole numbers, and some of them have decimals that carry on.

- **Are the numbers always getting bigger? Are they always getting smaller?**

No - it starts with 1, then 2, then it goes back down to 1.5, and it keeps going up and down. But does this sequence have a limit?

It looks like the numbers are getting closer and closer to a particular number - even if you don't know exactly, what do you think the number might be? Could it be bigger than 1.617 and smaller than 1.619?

The limit of this sequence is a special number, with a special name - it's called the **Golden Ratio**. It's 1.618033, with a decimal that carries on, and it has some interesting properties. We use this letter, which is a Greek letter called phi, to represent the Golden Ratio.

- **Practise writing the letter phi a few times**.

Here's one way in which the Golden Ratio is a cool number. Take your calculator, clear the screen, and press 1, then press divide, then type in the Golden ratio from this slide. Now press equals, and you should get an answer on your calculator. Now add 1 and press equals again. You should find you get back to phi! **That's interesting!**

Try this with a different number - put 1, divide by something, then add 1. will you get back to the same thing you divided by? This is something special that only works for the Golden ratio, phi. It's the only number this works for.

The Golden Ratio also sometimes appears in particular shapes:

This shape is a **five-pointed star**. It's a nice shape! If we measure some of the distances in this star - **A**, the distance all the way across it; **B**, the distance from the end to the far side of the base of a point; **C**, the distance to the near side; and **D**, the width of the base of the point. We can now work out some ratios with these numbers. It turns out that if we find A/B, and B/C, and C/D, these all equal the same number. Can anyone guess what number it is? The Golden Ratio! Correct, you get a gold star (and really, all stars are Golden, it turns out!)

This shape is called a **Rhombic Triacontahedron**. Isn't it great? It also contains the Golden Ratio. Because to make a Rhombic Triacontahedron, you put together pieces like this - does anyone know what this shape is called? It is a diamond shape, but it has a mathematical name too - it's called a **rhombus**. (I guess that's a bit obvious when you remember this is a Rhombic Triacontahedron). If you want all these pieces to fit together into a full shape, you need a very particular shape of rhombus: you need one for which the width of the rhombus, **A**, divided by the height, **B**, is… guess what? The Golden ratio!

If you divide a line into two parts like this - let's call this part **A** and this part **B** - so what's the length of the whole line? It's **A + B**. If I divide the whole line by **A**, that's **A+B/A**; if we choose our lengths carefully, then we could pick them so that **A+B/A** is the same as **A/B**. If we do, then that ratio is also the Golden Ratio, and again that's the only ratio that works this way. Isn't it nifty? 50 – 56

So this interesting ratio has lots of nice properties, and it appears out of nowhere when all you were trying to do was build a Rhombic Triacontahedron! Next we're going to explore another aspect of it - and we're going to use it to investigate nature.

The Golden Angle (20 minutes)

Now we're going to be working with angles, and dividing up this circle into parts. 57

If you divide a circle into 6 parts, you get six sections which are each ⅙ of a circle. 58

- **Can anyone tell me how many degrees each of these angles will be?** 59

Since a circle is 360 degrees, 360 ÷ 6 = 60 degrees in each part. 60 61

If we wanted to divide a circle into a number of parts that's not a whole number, could we do that? If we are dividing into 6 parts, we just divide by 6 - so if I wanted to divide it into three-and-a-half parts, I could still do that - I'd just divide 360 degrees by 3.5.

The Golden Ratio, phi, is about 1.618. 62

- What would we get if we divided a circle into the Golden Ratio parts? **Use your calculator now to work it out.** 63

If you divide 360 by 1.618 you get about 222.5 degrees. This is the angle here - but it's not the small part! The angle 222.5 degrees is this large part on the outside of the angle, because 222.5 is more than 180 degrees. 64 65 66

If we wanted a smaller angle, that's less than 180 degrees, we could use this inside angle here. This kind of angle is easier to work with. 67

How could we work out what this inside angle is, if the outside angle is 222.5? Work it out with your calculator.

If we want the inside angle, we can find 360 - 222.5 = 137.5 degrees. This angle is called the **Golden Angle**, because it's made using the Golden Ratio. It's a very special angle, and we're going to investigate why - and why some angles are more special than others, by making a picture of a flower.

This worksheet has a set of different angles on it, and you'll need to cut out each of these pieces carefully with a pair of scissors.

Now here's a worksheet which has a terrible flower on it - it's only got one flower petal!

If I want to make more petals, I can draw them on - but I'm going to make sure the petals are spaced out around the flower, using these angles. To start with, use the 60 degree angle.

Put your angle so it lines up with the centre of the flower, and the tip of the petal - and mark where the other side of the angle lies.

You can now draw a flower petal there - it may be easier to make a small mark and then take the angle away, so you can draw a petal more easily. Then for the next petal, do the same thing again: put the point of the angle at the centre, and measure the same distance round, to draw another petal.

You'll need to repeat this until you find you are drawing a petal in the same place you have already drawn a petal. **How many petals do you think you'll need to draw with this 60 degree angle before you do this?** Try it and check.

Then you can do the same with each of the other angles, and see what happens. If you find you're drawing a petal that's not quite in the same place as another petal, but it overlaps, you can draw the new petal behind the first one (or you can just overlap them if you prefer). Keep adding petals until you get back to where you started.

Here are some pictures of all the finished flowers - don't worry if you didn't manage to finish them all. Which angles do you think make a better flower? Do some angles work better than others?

Angles which are a simple fraction of the circle - like ¼ circle (90 degrees), or ⅙ circle (60 degrees) will give boring flowers, where the petals clump in separate groups. Even these flowers for 70 and 150 degrees, which look like they have lots of petals, are a bit boring - because they'll never get any more petals than this. The most interesting flowers are those where the petals are spread out nicely around the flower, and the Golden Ratio is the best angle to make this happen - you can keep drawing petals for much longer than we have done, and they will never lie on top of an existing petal.

A similar process is used when flowers position the seeds inside their seed head - they will place a seed, then turn through an angle, and then place more seeds, and you get the same patterns. This tool will allow us to try different angles and see what happens. The Golden Angle is the best one to use here too - it means the seeds are nicely packed in a spiral pattern. If this pattern looks familiar, it's because it's the way they are arranged in a sunflower head - using the Golden Angle!

This is not a coincidence - it's the best angle the flower can use for placing its seeds. This is one of the many reasons why it's such a beautiful angle, and why the Golden Ratio is such a wonderful number.

Counting Plants (20 minutes)

So we've seen how the Fibonacci numbers and the Golden Ratio are connected, and it turns out the Fibonacci numbers often crop up in nature as well. We're going to look at some real flowers now, and see another way in which Fibonacci numbers come in. [We've picked/you're going to pick] some particular types of flowers and we're going to count their petals. We're also going to look at pine cones - these aren't flowers, but they are grown in a similar way - by placing individual petals round a circle.

If you have a flower, you should first look at the way the petals are arranged around the circle. Does it look like a regular angle, or something more like the Golden Angle we used?

Now you should **count the petals** on one flower head. You might find it's easiest to do this by pulling them off the flower and making piles of five, or ten petals, especially if they're quite small.

If you have a pine cone, you need to **count the spirals** around the body of the pine cone. It might make it easier if you use a pencil or a marker pen to draw a line along one diagonal row of segments, then you can count how many rows you can see before you get back to your line. Once you've done this, try finding a diagonal running the other way and do the same.

Once you all have your numbers, **come and write them on this board** - let's see if there are any patterns. I'm going to circle all the numbers which are

Fibonacci numbers - we should find that many, but not all of these numbers, are Fibonacci numbers.

Why do you think might there be a Fibonacci number of petals? The seeds in the centre will be arranged in spirals like the sunflower, and each line of seeds will generate a petal at the end - so if there are a Fibonacci number of spirals, like on the pine cone, you'll get a Fibonacci number of petals.

Why might this not happen for every single plant/flower/pine cone? These are natural objects, generated by natural processes of growth. Some plants have minor mutations that make them slightly different from others; there may have been unusual weather conditions, or animals might have nibbled on them, just at a time when these plants were starting to form - which might change the shape of the flower or cone.

We're going to finish by looking at one more plant - does anyone recognise this? It's a pineapple. The pineapple is a fruit that grows in tropical regions, and it often also contains Fibonacci numbers. If we mark this spiral with a piece of coloured tape, we can count how many stripes run in each direction. Is this a Fibonacci number as well?

Fibonacci Spiral Jigsaw (15 minutes)

To finish today's workshop we're going to work on one more activity, in groups. Each group will be given a copy of these worksheets to cut out the pieces.

Each circle is drawn inside a square, and it can be cut into four separate pieces. If you do this, you should find you have smaller squares, each with a quarter of a circle in, and across the worksheets they are in all different sizes. There are two of the smallest one, and then one each of the five other sizes.

If you work in groups of four, you can **cut the pieces out** and share them so you each get a quarter of each circle. Share the work of cutting out, so nobody is waiting for anyone else, and make sure you all have the same sizes of pieces.

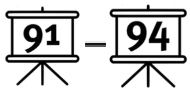
Now you can use this sheet, which has an empty rectangle on it. You need to join up the lines of the circles, so that they form one long curve - you might want to start with the biggest one to make sure you can fit them all in!

Arrange the pieces first, then once you're happy that your circle pieces all form one long spiral, you can **glue them down**. Do you like the shape? Here's a spiral made from these pieces.

This spiral is called the **Golden Spiral**, and it's based on the Fibonacci numbers. The sizes of the squares are 1, 1, 2, 3, 5, 8 and 13. Can you see that each pair of Fibonacci numbers adds up to the next one, so you can fit the two ones alongside the two, then the three fits alongside a 1 and a 2, and the 5 fits with the 2 & 3, and so on?

The Golden Spiral is an example of a **logarithmic spiral** - this means that the amount that the spiral curves by changes as you go along the curve. Spirals with this property occur elsewhere in nature - in the shapes of spiral galaxies, and some seashells - but you should remember that not all logarithmic spirals are Golden Spirals. The spiral on a seashell isn't a Golden Spiral!

End of session - recap



In this session we've looked at sequences of numbers, and played a game to see if they're getting closer and closer to a particular limit; we've looked at the Fibonacci sequence, and how you can use it to make ratios that get closer and closer to a special limit, called the Golden Ratio; and we've looked at how the Golden Angle and the Golden Spiral, and the Fibonacci numbers, sometimes appear in different places in the natural world.

So when you're next out in the forest and you see flowers and pine cones, don't forget that even outside of a maths classroom, mathematics is still there, hidden inside the things around us - and it means the flowers do their job as well as they possibly can!