Numbers of Nature Masterclass: Extra Background Information

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Sequences

A sequence, in mathematics, is a list of objects, like numbers, that follow a particular pattern. The individual elements in a sequence are called **terms**. A sequence may contain the same number twice, and it doesn't always have to carry on getting bigger or smaller. The order of the terms in the sequence is important, and the sequence can be finite (finish after a certain number of terms) or go on forever. Sequences are useful in a number of mathematical disciplines for studying patterns, shapes, and other mathematical structures.

In this session, we discuss sequences of numbers that do go on forever, and what happens if as it goes on it gets closer and closer to a single value. This is called the **limit** of the sequence, and is a very important concept in mathematics.

Limits of sequences

In the session, we give a few examples of sequences, and discuss what their limits might be.

Some sequences approach the limit from above, such as the sequence of numbers 1/n, where n = 1, 2, 3, 4, 5…

1 ½ ⅓ ¼ ⅕ …

These numbers are always getting smaller, and as the size of n increases they will get closer and closer to 0. The limit of this sequence is 0.

Some sequences approach their limit from below, by always increasing (a simple example would be -1/n, for n = 1, 2, 3, 4, 5…), while others change from bigger to smaller, but always getting closer to the limit. For example, the sequence $(-1)^n/n$ for n = 1, 2, 3, 4, 5... would give the numbers:

-1 ½ -⅓ ¼ -⅕ …

This is called an alternating sequence, because it alternates between positive and negative values.

If you already have a sequence with a given limit, it's possible to create a sequence for another limit, by adding the same amount to each term. For example, the sequence $5 + (1/n)$, for n = 1, 2, 3, 4, 5… has a limit of 5.

In the workshop, we play a game called **How Far Do You Have to Go?** This involves choosing a sequence with a limit, giving your friend a small (positive) number, and challenging them to work out how far along the sequence you need to go before they are closer to the limit than that number. If the limit of the sequence is 0, and the sequence is approaching 0 from above, this is the same as working out how far along the sequence you need to go before the term is smaller than your given small number. If the limit isn't 0, or it's approaching it in a different way, you'll need to check the difference between the term of the sequence and the limit, and go far enough so that the difference is smaller.

Mathematicians often study sequences of numbers like this, and this game is a very useful way to check whether a sequence has a limit. This idea is called **convergence**, and the formal definition says:

> A sequence, whose terms are written S_1, S_2 and so on, **converges** to a given limit **c** if for any small number you name (in formal definitions this number is often given the symbol **epsilon**, written ε), there is a number N such that the difference between the Nth term of the sequence and **c** is less than ε, which we can write as:
>
> $|S_N - \mathbf{c}| < \varepsilon$
>
> and that this difference will continue to be less than ε for all the terms after that.

The vertical lines (pipes) in the inequality above indicate that if $S_N - c$ is negative (as we don't know if the sequence is approaching from above, or below, or alternating, so it might be that $S_N - c$ is less than zero) then you ignore the minus sign - it's called the **absolute difference** between the two numbers.

In the examples given, some of the sequences we discuss have a very simple rule for the terms of the sequence - they're called the **constant sequence**, and one of the examples in the workshop is the sequence 2, 2, 2, 2, 2, … which is always 2. The limit of this sequence is 2, and with this definition of convergence you can see that the difference between each term and the limit is always 0, which will be less than any small number you can name.

There are many sequences which don't get closer and closer to a limit. For example, the sequence n^2, for n = 1, 2, 3, 4, 5… will get bigger each time, but the terms are also getting further apart, and not getting closer to any one number. This kind of sequence doesn't have a limit, and we say it **diverges**.

You could argue that if you look at the terms of the sequence, the numbers are getting closer to the number 100. This works for the first 10 terms, but once you get beyond that, they will no longer be getting closer to 100. This is why our definition of a sequence converging needs you to find a number N for which all the terms beyond that are closer to the limit.

Some people might suggest that if a sequence continues to get bigger, it has a limit at infinity. This makes sense, but mathematically it's not a correct statement. A limit has to be a number, and infinity is not a number!

Fibonacci Numbers

One sequence that's studied in this workshop is the Fibonacci sequence - this is defined using the rule that each entry in the sequence is the sum of the two previous entries. This means to define the sequence completely, the first two terms must be given, and in the workshop we use 1, 1 as the first two terms.

Fibonacci numbers occur in many places - as can be seen in the Rabbits and Sequences workshop, calculating the number of ways to arrange 1p and 2p coins produces Fibonacci numbers, as does a simple model for rabbit populations. They're also connected to Pascal's triangle. They're so common in mathematics, a whole journal called Fibonacci Quarterly is dedicated to new discoveries connected to Fibonacci numbers. They also crop up in divisibility algorithms, project planning methods, search algorithms, and generating random numbers.

The Fibonacci sequence is named after Leonardo de Fibonacci, who lived approximately 1175 – 1250, and was sometimes called Leonardo of Pisa. While Fibonacci wasn't the first person to discover this sequence, he was the first to introduce the idea to Western mathematics, in his book in 1202, which was called Liber Abaci ('Book of Calculation'). In this book he also popularised the system of Hindu-Arabic numerals we use today, including the use of the digit 0 and place value in decimals.

The Golden Ratio

The Fibonacci sequence doesn't converge to a limit, as its terms keep getting bigger and further apart. But it is possible to create a sequence from the Fibonacci sequence which does have a limit, by taking successive pairs of terms and dividing the smaller by the larger.

Dividing one number by another like this gives a ratio, and using the terms of the Fibonacci sequence we can create a long sequence of ratios, and observe that this new sequence we've created does converge, and it has a limit.

N	Nth Fibonacci number	(N+1)th Fibonacci number	(N+1)th divided by Nth
1	1	1	1/1 = 1
2	1	2	2/1 = 2
3	2	3	3/2 = 1.5
4	3	5	5/3 = 1.66…
5	5	8	8/5 = 1.6
6	8	13	13/8 = 1.625
7	13	21	21/13 = 1.615…
8	21	34	34/21 = 1.619…
9	34	55	55/34 = 1.617…

The limit of this sequence is called the Golden Ratio, written using the symbol Φ (phi) which has a value of around 1.618. In the table, it can be seen that the sequence of ratios approaches this limit from above and below - the number moves up and down either side of Φ, and gets closer each time.

Φ can also be written as $(\sqrt{5}+1)/2$, and it has various interesting properties, as discussed in the workshop:

- The Golden Ratio is the unique number for which $1/\Phi + 1 = \Phi$
- If you divide a line into two parts such that the ratio of the small part to the large part is the same as the ratio of the larger part to the whole line, it will be in the Golden ratio
- The Golden ratio is present in the proportions of a five-pointed star - in particular, one which has been constructed using lines which join the points of a pentagon. Some five-pointed stars have wider points, and it will not work unless the line running from a point to the body of the star runs straight across and joins straight on to the opposite line
- The Golden Ratio allows you to construct a **rhombic triacontahedron**: this is a shape with 30 faces, each of which is a rhombus whose diagonals (the two distances across the rhombus from corner to corner) are in the Golden ratio. This kind of rhombus is sometimes also called a Golden Rhombus.

Since the ratio between one mile and one kilometre is close to the Golden ratio, this means that if you know a distance in miles and want to find it in kilometres, and your number of miles is in the Fibonacci sequence, the number of kilometres will be roughly the next Fibonacci number - for example, 3 miles is around 5km, and 8 miles is around 13km.

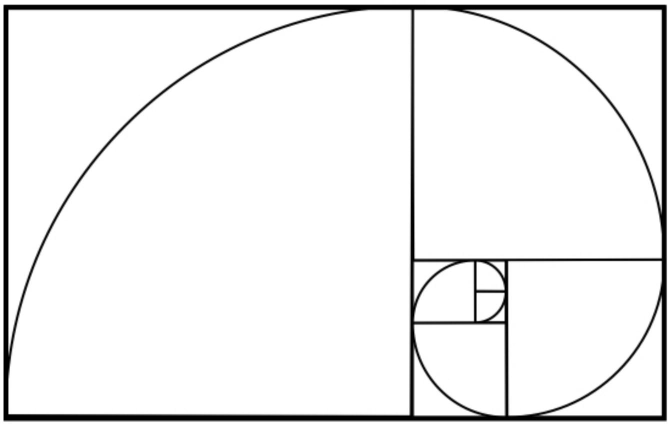
The Golden Angle

In the workshop, we consider what the Golden ratio represents as an angle. Dividing a whole circle (360 degrees) into a number of parts gives a proportion of a circle - for example, dividing by 6 gives 6 equal parts, each measuring 60 degrees. If we wanted a Golden angle, we could divide 360 by our Golden ratio, and doing so gives 360/1.618… = 222.5 degrees (approximately). This, and its **complementary angle** 360 - 222.5 = 137.5 degrees, are both special angles related to the Golden ratio.

It makes sense to calculate the smaller angle, as 222.5 degrees is an angle larger than 180 degrees (this is called a **reflex angle**) so working with its complementary angle is easier, and is the same as measuring an angle of 222.5 degrees but going round the other way.

This angle is found in the way petals are spaced around certain types of flowers, and seeds are positioned inside flower heads.

The Golden Spiral



Another activity in the workshop involves constructing a Golden Spiral, also called a Fibonacci spiral. This is made using square pieces each of which has a quarter of a circle inside, and the length of the side of the square is a Fibonacci number.

These squares can be assembled into a spiral, such that the curves all match up and create a continuous curve around the shape. This spiral can continue to be built forever, since each Fibonacci number is the sum of the previous two, and the squares are arranged such that the next square will always line up with the edges of the next two smallest squares - so in the diagram here, another square could be added across the bottom, then another on the right, and so on.

If you build up the Fibonacci spiral from the smallest pieces upwards, the outside rectangle of the spiral at each stage has sides which are two consecutive Fibonacci numbers, starting with 2 by 1, then 3 by 2, then 5 by 3 and so on. These are exactly the pairs from our table above, and if you calculate the ratio between them, this specifies the shape of the rectangle. The ratio between the sides of the rectangle is getting closer and closer to the Golden ratio, and if you made a rectangle with exactly this ratio it's called a Golden Rectangle. This rectangle is sometimes used in design - for example, standard sized credit cards and business cards are often a Golden rectangle.

Because of the way the Golden rectangle is defined, it has the property that if you cut a square off one end of the rectangle, the piece left behind is also a Golden rectangle. This means if you place one rectangle horizontally against one oriented vertically, the diagonal line between the corners of the horizontal rectangle will continue and hit the corner of the vertical rectangle, as that top part is in the same ratio.

Numbers in Nature

Fibonacci numbers occur in nature - as seen in the workshop, counting the number of spirals on a pine cone or sunflower seed head will result in a Fibonacci number.

This is because the spacing of the seeds is optimal when it's least likely to overlap with itself, so the Golden ratio is useful in making the gap between seeds least likely to be a fraction of a whole turn - if the seeds were placed ¼ turn apart, every fourth seed would overlap, but with seeds 1/Φ of a turn apart (our Golden angle), it will be a long time before a seed lands in the same place, and this means the number of turns will be a Fibonacci number.

Of course, every plant is different, and natural variation means this sometimes doesn't work - some plants grow in different circumstances, or their growth could be interrupted or changed by physical interference, random mutations and weather conditions.