Christmas Lectures 2006

Stable and unstable balances

INTRODUCTION

The word chaos is sometimes used to describe a situation where events are happening that are beyond control or even reason – where anything could happen and nothing is predictable. Mathematical chaos is different and, at first, the difference may seem quite subtle.

EXPERIMENT ONE: BALANCING A PENCIL

Take a sharp pencil and try to balance it on its point on a flat surface. How long can you get it to stand up straight for? Can you predict which way it will fall?

With a lot of practice and a steady hand it is possible to get the pencil to balance for a few seconds. It is really difficult to get it to balance for longer. Why do you think this is?

The pencil balances when there is an equal weight either side of the point, but it's no good if the weight is approximately equal. Any slight change away from the exact balance will make the pencil topple.

This experiment shows two of the characteristics that are common to mathematical chaos. When a pencil, for example, is balanced we see that:

1. Sudden and dramatic changes may occur at any point.
2. The nature of these changes is very sensitive to small effects.

The sudden change occurs when the pencil begins to fall over, and the small effects that cause the change can be anything from a tiny vibration in the table to a light draught from a door being opened. By 'the nature of the change' we mean which way and how fast the pencil falls. Once the pencil starts falling in one direction, it continues; but the initial direction is influenced by things that are very difficult to measure.

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BUTTERFLIES AND HURRICANES

Probably the most well-known example of a chaotic system is the weather. As you know, the weather often goes through dramatic changes and it's hard to predict which way it will go. In fact you can probably think of lots times when the weather has suddenly changed from sunny to stormy, or when the weather forecast on the television has been completely wrong.

So why is the weather so hard to predict?

One reason it is so hard to predict is that within the weather there are lots of systems that are similar to the balanced pencil. Whether or not a cloud (heavy with water) will start to rain, whether or not the wind will change direction, whether a tornado will form or not … these are all things that can change suddenly.

This is not to say that the weather is theoretically unpredictable: if you knew everything about the weather today you would know everything about it tomorrow. Every future change in the weather is caused by a chain of events that are already in process right now. The problem is that tiny differences today can grow to have larger effects tomorrow. For example, a butterfly flapping its wings in China might cause a hurricane to start in Brazil – rather like someone sneezing in the next room may cause your pencil to fall over.

Of course, it is not just the butterfly that starts the hurricane: there are millions of different causes for weather patterns. Some are large and predictable (such as the change in seasons), others are tiny and usually don't affect the weather much. And sometimes a tiny change can grow in effect and tip the weather from one state into another. But the question can still be asked: what makes the weather have this property? (Why is the weather more like a balanced pencil than, for example, balanced scales?)

The answer lies in the special properties of high-speed winds, which you can see by trying this experiment.

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EXPERIMENT TWO: WIND SPEEDS

High and low wind speeds have different properties. To test this take a flag (or a strip of paper or fabric) and walk in a straight line holding the flag out so that it trails behind you. It should just float along.

Now try doing the same thing at a much faster speed, either by holding the flag out of a car window or by running as fast as you can into a strong wind. It should flap around wildly!

The reason is that wind going around an object such as a flag behaves differently at different speeds. At lower speeds the wind is easily predictable, whereas at higher speeds wind behaves much more chaotically.

What happened to the flag at the higher speed is a phenomenon that mathematicians call **turbulence**, and understanding this is the key to unlocking the behaviour of weather. You may have heard the word during a flight because pilots often blame turbulence for unsettling the aeroplane.

Turbulence can be seen in this picture of smoke rising. At the bottom the smoke is travelling at a low speed and rises in a single column. However, as it rises higher the smoke accelerates and its flow becomes turbulent. It starts to flow randomly from side to side, billowing into complex patterns and surprising shapes.

At the top and the bottom of the picture there is the outside influence of tiny air currents. At the bottom these don't change the shape of the smoke much, but at the top the smoke pattern is strongly influenced by these tiny changes.

Imagine this picture shows the air current across the Atlantic Ocean from North America to Europe, and you can start to imagine why the weather becomes difficult to predict.

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THE MILLION-DOLLAR QUESTION

The good news is that there are equations that describe the behaviour of the wind. Most weather forecasters use these equations to predict what is going to happen. The bad news is that nobody knows how to solve them! Forecasters can only use approximations. Although their approximations differ from the right answer by only a small amount, since the weather can be changed by tiny differences it means that their predictions can often be completely wrong.

These equations, called the **Navier-Stokes equations** after the two nineteenth-century mathematicians who discovered them, are not simple. A common representation of them is:

1) $$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t)$$

2) $$\text{div } u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0$$

If you don't understand some of the symbols in these equations, don't worry – not many people do!

The search for a solution to Navier-Stokes equations is a crucial part of our efforts to predict the weather. They are so important that there is a US$1 million prize on offer for the first person to solve them.

Visit the Clay Mathematics site (http://www.claymath.org/millennium/Navier-Stokes_Equations/) to find out more about the Million Dollar Prize for unlocking the secrets of the Navier-Stokes equations.