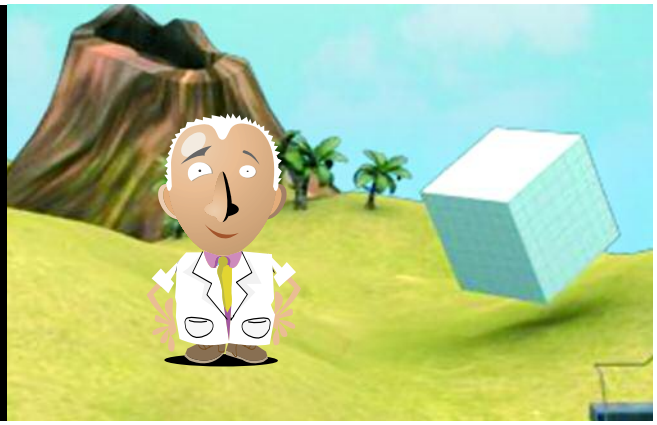
Christmas Lectures 2006

Fascinating fractals



Part two – infinite length

As Part one shows, we have to stop drawing fractals at some stage because we cannot, even with a computer, draw the smaller and smaller lines they require. But this doesn't stop a mathematician! Even if we can't explicitly draw a fractal indefinitely, we still have a very good idea of what it looks like, we can imagine it and we can talk about it.

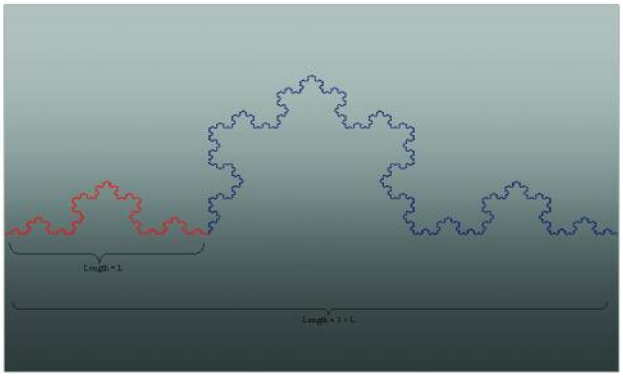
For example, we can ask questions about a fractal shape... like:

What is the length of this fractal?

... and the answers can be very surprising!

INVESTIGATING FRACTALS

We'll take the first fractal we made earlier. In this picture, part of the fractal is coloured red:



The red part is made by shrinking the whole fractal. Can you say by how much?

The distance from the left end to the right end of the red part is exactly one-third of the width of the whole fractal. If we say the length of the whole shape is X and the length of the red part is L, we've shown:

$$X = 3 \times L$$

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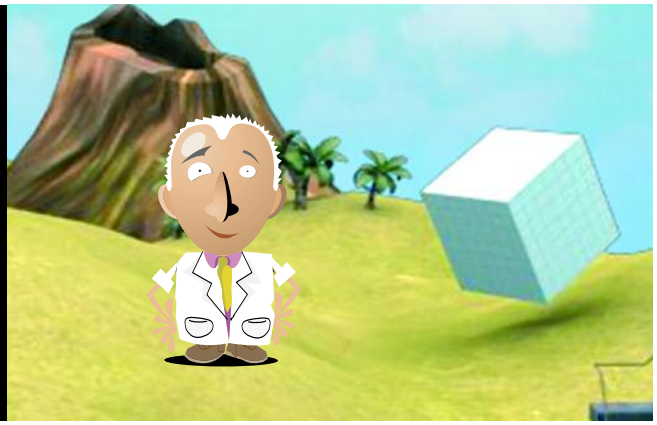
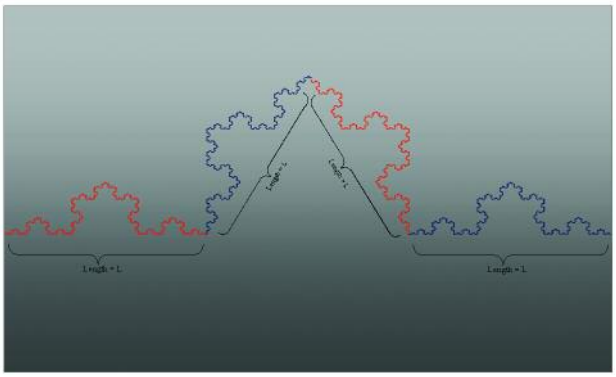


Now let's look at the fractal in a different way.

By rotating the red part around, we can see that it fits into the whole fractal exactly four times:

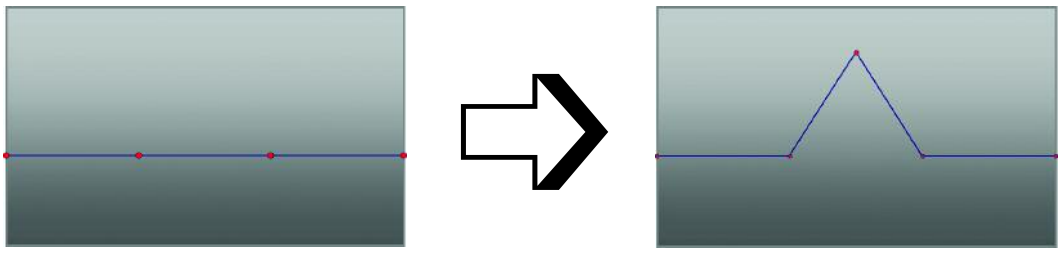


The length of the whole fractal must be four times the shape of the red part.
So we have:

$$X = 4 \times L$$

But we also have X = 3 x L. How can this be possible? Have we made a mistake? In fact, it is possible and we haven't made a mistake. Both equations are true because both X and L are equal to infinity.

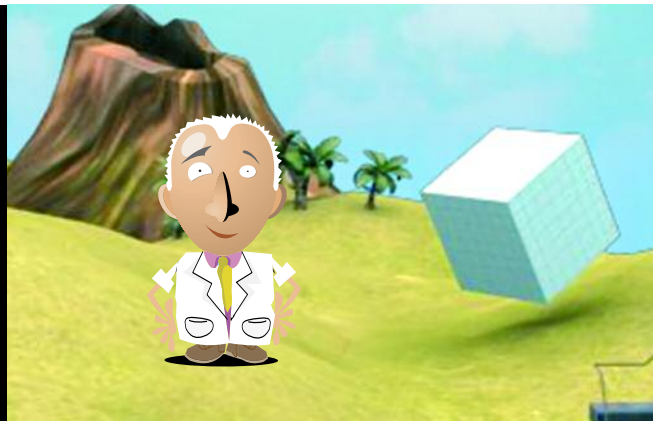
Still not convinced? Let's look at the fractal another way. Looking closely at it you can see that at each stage three lines of equal length are replaced by four lines of the same length:



Okay, you say, but what does this do to the length of the whole shape?

- The length of the left-hand picture is three times the length of the smaller line.
- The length of the right-hand picture is four times the length of the smaller line.

So the length of the right-hand picture is 4/3 times as long as the left-hand picture.



Since at each stage we do this to every line, we have:

At each stage, the length of the fractal grows by 4/3.

If we do this infinitely many times, we will increase the fractal's length infinitely many times. This proves:

The length of our fractal is infinite.